Construction of conjugate Harmonic function:

* If the real part u is given, then

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

* If the imaginary part v is given, then

$$u = \int \left[\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right]$$

1. Show that $u = y + e^x \cos y$ is harmonic and hence find its conjugate harmonic.

Soln.

Given $u = y + e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = 1 - e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y$$
$$= 0$$

Hence u satisfies laplace eqn.

$\therefore$ u is harmonic.

Now

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

$$= \int \left[-(1 - e^x \sin y) dx + e^x \cos y \, dy \right]$$

$$= \int -dx + \int e^x \sin y \, dx + \int e^x \cos y \, dy$$

$$= -x + e^x \sin y + e^x \sin y + C$$

$$v = 2e^x \sin y - x + C$$

2]. Show that $u = \cos x \cosh y$ is harmonic. Find its conjugate harmonic.

Soln.

Given $u = \cos x \cosh y$

$\frac{\partial u}{\partial x} = -\sin x \cosh y$ | $\frac{\partial u}{\partial y} = \cos x \sinh y$

$\frac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$ | $\frac{\partial^2 u}{\partial y^2} = \cos x \cosh y$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y$

$= 0$

Hence u satisfies laplace eqn.

$\therefore$ u is harmonic.

Now $v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$

$= \int \left[-\cos x \sinh y \, dx - \sin x \cosh y \, dy \right]$

$= -\sin x \sinh y - \sin x \sinh y + c$

$v = -2 \sin x \sinh y + c$

3]. Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic function. Find conjugate harmonic function

Soln.

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$ | $\frac{\partial^2 u}{\partial x^2} = 6x + 6$

$\frac{\partial u}{\partial y} = -6xy - 6y$ | $\frac{\partial^2 u}{\partial y^2} = -6x - 6$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6$$
$$= 0$$

Hence u satisfies laplace eqn.

$\therefore$ u is harmonic.

Now,

$$v = \int \left[-\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \right]$$
$$= \int \left[-(-6xy - 6y)\, dx + (3x^2 - 3y^2 + 6x)\, dy \right]$$
$$= \int (6xy + 6y)\, dx + (3x^2 - 3y^2 + 6x)\, dy$$
$$= \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy$$
$$v = 6x^2y + 12xy - y^3 + c$$