Laplace equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$ is called Laplace equation.

Harmonic equation:

Any function with 2 variables having its 2nd order partial derivatives which satisfies laplace eqn. is called a harmonic eqn.

Conjugate Harmonic function:

If u and v are harmonic functions such that u+iv is analytic, then each is called the conjugate harmonic function of the other.
Here u is conjugate harmonic of v and v is conjugate harmonic of u.

1]. prove that $u = e^x \cos y$ is harmonic.

Soln.

Given $u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

Hence u satisfies laplace equation.
$\therefore$ The function u is harmonic.

2]. prove that $u = \frac{1}{2} \log (x^2 + y^2)$ is harmonic.

Soln.

Given $u = \frac{1}{2} \log (x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \frac{1}{x^2+y^2} (2x)$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2} \frac{1}{x^2+y^2} (2y)$$

$$= \frac{y}{y^2+x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

Hence u satisfies laplace eqn.

$\therefore$ u is harmonic