



SNS COLLEGE OF TECHNOLOGY

(An Autonomous Institution)

Coimbatore-641035.



UNIT-II COMPLEX DIFFERENTIATION

Harmonic Conjugate

Construction of conjugate Harmonic function:

* If the real part u is given, then

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

* If the imaginary part v is given, then

$$u = \int \left[\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right]$$

Q. Show that $u = y + e^x \cos y$ is harmonic and hence find its conjugate harmonic.

Soln.

Given $u = y + e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \left| \frac{\partial u}{\partial y} = 1 - e^x \sin y \right.$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \left| \frac{\partial^2 u}{\partial y^2} = -e^x \cos y \right.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y \\ = 0$$

Hence u satisfies Laplace eqn.
 $\therefore u$ is harmonic.

Now

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

$$= \int \left[-(1 - e^x \sin y) dx + e^x \cos y dy \right]$$

$$= \int -dx + \int e^x \sin y dx + \int e^x \cos y dy$$

$$= -x + e^x \sin y + e^x \cos y + C$$

$$v = 2e^x \sin y - x + C$$



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Q. Show that $u = \cos ax \cosh by$ is harmonic. Find its conjugate harmonic.

Soln.

Given $u = \cos ax \cosh by$

$$\frac{\partial u}{\partial x} = -\sin ax \cosh by \quad \left| \begin{array}{l} \frac{\partial u}{\partial x} = -\cos ax \sin by \\ \frac{\partial u}{\partial y} = \cos ax \cosh by \end{array} \right.$$

$$\frac{\partial^2 u}{\partial x^2} = -\cos ax \cosh by \quad \left| \begin{array}{l} \frac{\partial u}{\partial x} = -\cos ax \cosh by \\ \frac{\partial u}{\partial y} = \cos ax \cosh by \end{array} \right.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\cos ax \cosh by + \cos ax \cosh by = 0$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic.

Now

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right]$$

$$= \int \left[-\cos ax \sin by dx - \sin ax \cosh by dy \right]$$

$$= -\sin ax \sin by - \cos ax \sin by dx + C$$

$$v = -2 \sin ax \sin by + C$$

Q. Prove that $u = x^3 - 3xy^2 + 3x^2 - 3y^2$ is harmonic function. Find conjugate harmonic function.

Soln.

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \quad \left| \begin{array}{l} \frac{\partial^2 u}{\partial x^2} = 6x + 6 \\ \frac{\partial^2 u}{\partial y^2} = -6x - 6 \end{array} \right.$$

$$\frac{\partial u}{\partial y} = -6xy - 6y \quad \left| \begin{array}{l} \frac{\partial^2 u}{\partial x^2} = 6x + 6 \\ \frac{\partial^2 u}{\partial y^2} = -6x - 6 \end{array} \right.$$



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$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + 6 - 6x - 6 \\ = 0$$

Hence u satisfies Laplace eqn.

$\therefore u$ is harmonic.

Now,

$$v = \int \left[-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right] \\ = \int [(-6xy - 6y) dx + (3x^2 - 3y^2 + 6x) dy] \\ = \int (6xy + 6y) dx + (3x^2 - 3y^2 + 6x) dy \\ = \frac{6x^2y}{2} + 6xy + 3x^2y - \frac{3y^3}{3} + 6xy \\ v = 6x^2y + 12xy - y^3 + C$$