



Mobius Transformation (or) Bilinear Transformation

The transformation  $w = \frac{az+b}{cz+d}$ ,  $ad-bc \neq 0$ , where  $a, b, c, d$  are complex numbers, is called a bilinear transformation.

Formula:

Bilinear transformation of  $z_1, z_2, z_3$  into  $w_1, w_2, w_3$  is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Q. Find the bilinear transformation which maps the points  $z=0, -i, -1$  into  $w=i, 1, 0$  respectively.

Soln.:

Given  $z_1=0, z_2=-i, z_3=-1$

$w_1=i, w_2=1, w_3=0$

The bilinear transformation is,

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w-wi} = \frac{z(1-i)}{(-i)(z+1)}$$

$$\frac{w-i}{w-wi} = \frac{z-zi}{-iz-i}$$

$$(w-i)(-iz-i) = (z-zi)(w-wi)$$

$$-wzi - wi - z - 1 = wz - wzi - wzi - wz$$

$$-wzi - wi + wzi + wz = z + 1$$

$$wzi - wi = z + 1$$

$$wi(z-1) = z + 1$$



$$\omega = \frac{1}{i} \frac{z+i}{z-1}$$

$$= \frac{i}{i^2} \frac{z+i}{z-1}$$

$$\omega = -i \left( \frac{z+i}{z-1} \right)$$

2]. Find the bilinear transformation which maps  $z = 1, i, -1$  to  $w = i, 0, -i$ .

Soln.

Given

$$z_1 = 1, \quad z_2 = i, \quad z_3 = -1$$

$$w_1 = i, \quad w_2 = 0, \quad w_3 = -i$$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \quad \text{(or)} \quad \frac{i(w-i) + (-i)(w+i)}{i(w-i) - (-i)(w+i)}$$

$$\frac{(w-i)}{-(w+i)} = \frac{z-1}{z+1} \left( \frac{1+i}{-1+i} \right)$$

$$= \left( \frac{z-1}{z+1} \right) \frac{(1+i)(-1-i)}{(-1+i)(-1-i)}$$

$$= \left( \frac{z-1}{z+1} \right) \frac{-1-i-i+1}{(-1)^2 - i^2}$$

$$= \frac{z-1}{z+1} \left( \frac{-2i}{2} \right)$$

$$-\left[ \frac{w-i}{w+i} \right] = -\left[ \frac{z-1}{z+1} \right] i$$

$$\frac{w-i}{w+i} = \frac{i(z-1)}{z+1}$$

By componendo & dividendo,

$$\frac{(w-i) + (w+i)}{(w-i) - (w+i)} = \frac{(i(z-1) + (z+1))}{(i(z-1) - (z+1))}$$

$$w = \frac{-iz-1}{iz-1}$$



$$\Rightarrow \frac{2w}{-2i} = \frac{z(i+1) + (1-i)}{z(i-1) - (1+i)}$$

$$w = -i \frac{z(i+1) + (1-i)}{z(i-1) - (1+i)}$$

$$w = \frac{-zi - i}{-i + 1}$$

Find the bilinear transformation which maps  $-1, -i, 1$  in  $z$ -plane to  $\infty, i, 0$  in  $w$ -plane respectively.

Soln.

Given  $z_1 = -1, z_2 = -i, z_3 = 1$

$w_1 = \infty, w_2 = i, w_3 = 0$

The Bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$w_1 = \left(\frac{w}{w_1} - 1\right)(w_2 - w_3) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(0-1)(i-0)}{w(0-1)} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$\frac{-1}{-w} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$w = i \frac{(z-1)(-i+1)}{(z+1)(-i-1)}$$

$$= i \frac{[-iz + z + i - 1]}{(-iz - z - i - 1)}$$

$$= \frac{z + iz - 1 - i}{-iz - z - i - 1}$$

$$= \frac{(z-1) + i(z-1)}{(-z-1) + i(-z-1)}$$



$$= \frac{(z-1)(1+i)}{(-z-1)(1+i)}$$

$$w = \frac{1-z}{1+z}$$

$$w = \frac{1-z}{z+1}$$

4]. Find the bilinear transformation which maps  $\infty, i, 0$  to  $0, i, \infty$ .

Soln.

Given  $z_1 = \infty, z_2 = i, z_3 = 0$

$w_1 = 0, w_2 = i, w_3 = \infty$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(i-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$w = \frac{1}{z}$$



Soln.

$$\text{Given } z_1 = 0, z_2 = 1, z_3 = \infty$$

$$w_1 = -5, w_2 = -1, w_3 = 3$$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)z_3 \left( \frac{z_2}{z_3} - 1 \right)}{z_3 \left( \frac{z}{z_3} - 1 \right) (z_2-z_1)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-1+5)} = \frac{(z-0)(0-1)}{(w-1)(1-0)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{-z}{-1}$$

$$\frac{-(w+5)}{w-3} = \frac{z}{1}$$

$$-(w+5) = z(w-3)$$

$$-w-5 = wz-3z$$

$$wz+w = 3z-5$$

$$w(z+1) = 3z-5$$

$$w = \frac{3z-5}{z+1}$$