Standard Transformation:

TYPE - I

I. Translation $[w = c + z]$

problem:

1. What is the region of the w-plane into which the rectangular region in the z-plane bounded by the lines $x = 0, y = 0, x = 1$ and $y = 2$ is mapped under the transformation $w = z + (2 - i)$?

Soln.

Given $w = z + (2 - i)$

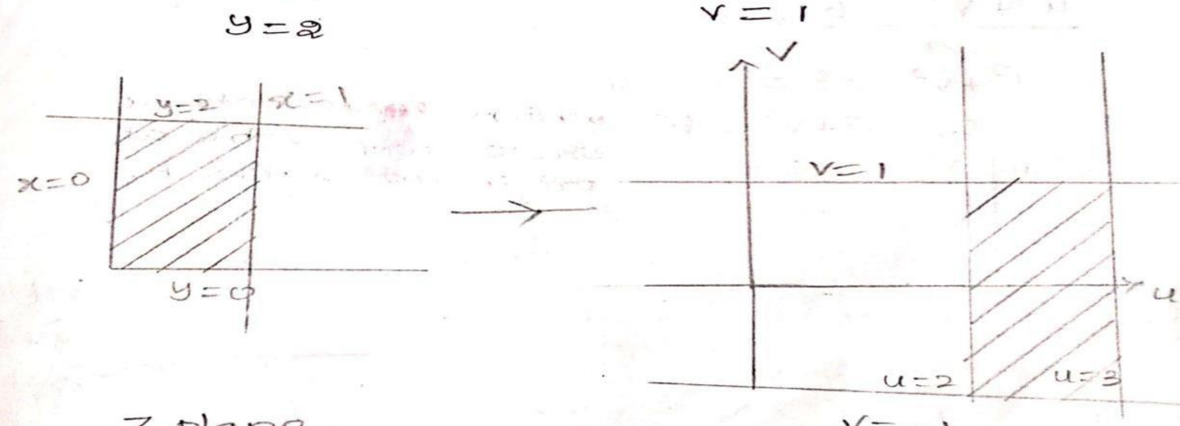
$u + iv = x + iy + (2 - i)$

$u + iv = (x + 2) + i(y - 1)$

Equating real & imaginary parts,

$u = x + 2$ and $v = y - 1$

Boundary lines	Transformed boundary lines
$x = 0$	$u = 2$
$y = 0$	$v = -1$
$x = 1$	$u = 3$
$y = 2$	$v = 1$



Z-plane w-plane

Hence the lines $x = 0, y = 0, x = 1$ & $y = 2$ in z-plane are transformed into $u = 2, v = -1, u = 3$ and $v = 1$ respectively in w-plane

Type 2:

Magnification and Rotation [$w = cz$]

1. Find the image of circle $|z| = 3$ under the transformation $w = 2z$

Soln.

Given $w = 2z$

$$u + iv = 2(x + iy)$$
$$u + iv = 2x + i2y$$

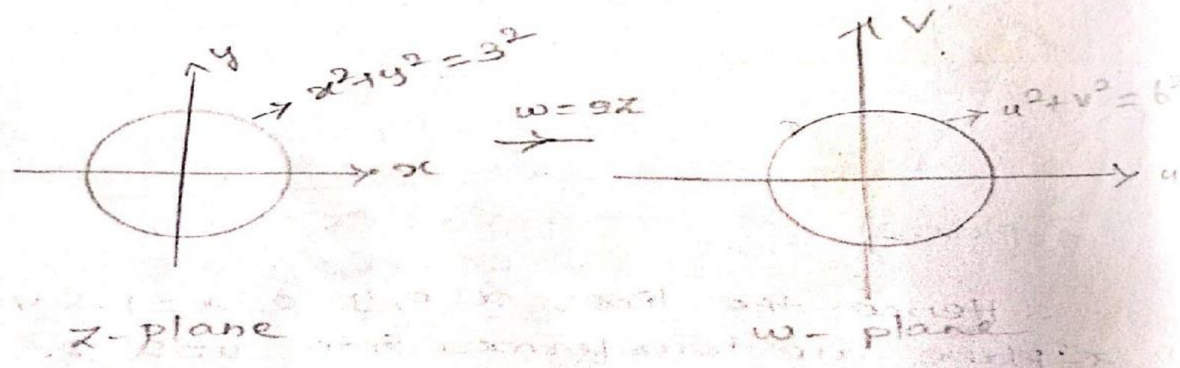
$\Rightarrow u = 2x$ and $v = 2y$

$x = \frac{u}{2}$ $y = \frac{v}{2}$

To find the image of $|z| = 3$.

$$|x + iy| = 3$$
$$\sqrt{x^2 + y^2} = 3$$
$$x^2 + y^2 = 3^2$$
$$x^2 + y^2 - 9 = 0$$
$$\left(\frac{u}{2}\right)^2 + \left(\frac{v}{2}\right)^2 - 9 = 0$$
$$\frac{u^2 + v^2 - 36}{4} = 0$$
$$u^2 + v^2 - 36 = 0$$

ie, $u^2 + v^2 = 6^2$ which represents a circle with centre at origin and radius 6.



2]. Find the image of the region $y > 1$ under $w = (1-i)z$

Soln.

Given $w = (1-i)z$

$$u + iv = (1-i)(x+iy)$$
$$= x + iy - ix + y$$
$$u + iv = (x+y) + i(y-x)$$
$$\Rightarrow u = x+y \quad \text{and} \quad v = y - x$$

Now,

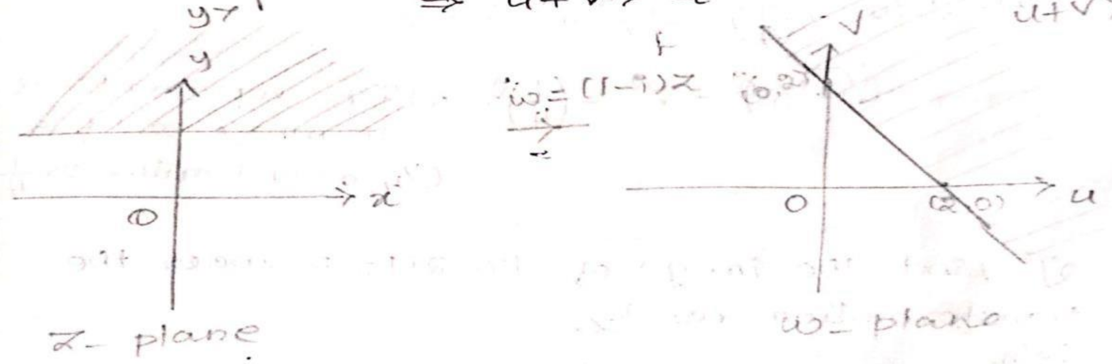
$$u + v = x + y + y - x$$
$$u + v = 2y$$
$$y = \frac{u+v}{2}$$

$$u - v = x + y - y + x$$
$$u - v = 2x$$
$$x = \frac{u-v}{2}$$

To find the image of $y > 1$

i.e., $\frac{u+v}{2} > 1$

$$\Rightarrow u + v > 2$$



TYPE 3 :

Reversion and Reflection [$w = \frac{1}{z}$]

1. Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$.

Soln.

Given $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\Rightarrow x = \frac{u}{u^2+v^2} \rightarrow (1) \quad \text{and} \quad y = -\frac{v}{u^2+v^2} \rightarrow (2)$$

when $x = 2$

$$(1) \Rightarrow 2 = \frac{u}{u^2+v^2}$$

$$2(u^2+v^2) = u$$

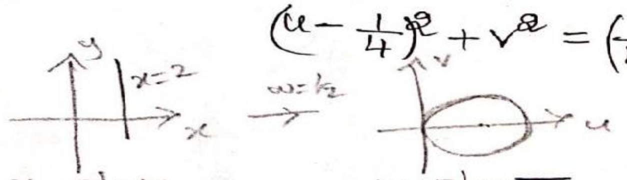
$$u^2+v^2 - \frac{1}{2}u = 0$$

$2ub = \frac{1}{2}u$

$b = \frac{1}{4}$

$$u^2 - \frac{1}{2}(u) + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 + v^2 = 0$$

$$\left(u-\frac{1}{4}\right)^2 + v^2 = \left(\frac{1}{4}\right)^2$$ which represents a circle whose centre is $(\frac{1}{4}, 0)$ and radius is $\frac{1}{4}$.



2]. Find the image of $|z-2i| = 2$ under the transformation $w = 1/z$.

Soln.

Given $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$\Rightarrow x = \dfrac{u}{u^2+v^2}$ and $y = \dfrac{-v}{u^2+v^2}$

To find the image of $|z-2i| = 2$

$|x+iy-2i| = 2$

$|x+i(y-2)| = 2$

$x^2+(y-2)^2 = 2^2$ which is a circle with centre $(0,2)$ & radius 2.

$x^2+y^2+4-4y-4 = 0$

$x^2+y^2-4y = 0$

$\left(\dfrac{u}{u^2+v^2}\right)^2 + \left(\dfrac{-v}{u^2+v^2}\right)^2 - 4\left(\dfrac{-v}{u^2+v^2}\right) = 0$

$\dfrac{u^2}{(u^2+v^2)^2} + \dfrac{v^2}{(u^2+v^2)^2} + \dfrac{4v}{u^2+v^2} = 0$

$\dfrac{u^2+v^2+4v(u^2+v^2)}{(u^2+v^2)^2} = 0$

$(u^2+v^2)+4v(u^2+v^2) = 0$

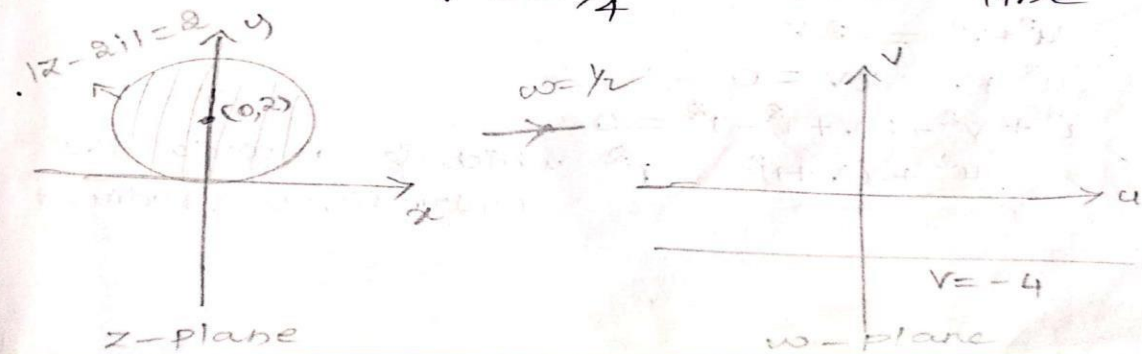
$\dfrac{(u^2+v^2)(1+4v)}{(u^2+v^2)^2} = 0$

$\dfrac{1+4v}{u^2+v^2} = 0$

$1+4v = 0$

$4v = -1$

$v = -\frac{1}{4}$ which is a straight line



3). Find the image of infinite strip i). $\frac{1}{4} < y < \frac{1}{2}$ ii). $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Soln.

Given $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x+iy = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$x+iy = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}$$

i). $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$, $\frac{-v}{u^2+v^2} = \frac{1}{4}$

$$u^2+v^2 = -4v$$

$$u^2+v^2+4v = 0$$

$$u^2+v^2+4v+2^2-2^2 = 0$$

$u^2+(v+2)^2 = 4$ which is a circle with centre $(0,-2)$ & radius 2.

when $y = \frac{1}{2}$,

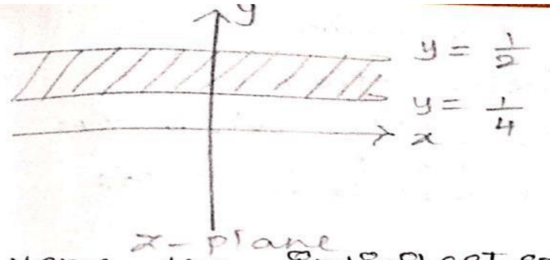
$$\frac{1}{2} = \frac{-v}{u^2+v^2}$$

$$u^2+v^2 = -2v$$

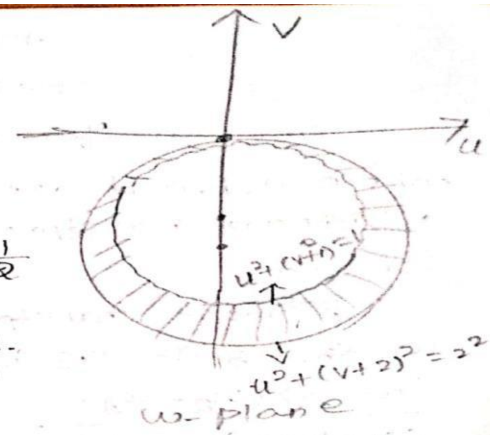
$$u^2+v^2+2v = 0$$

$$u^2+v^2+2v+1^2-1^2 = 0$$

$u^2+(v+1)^2 = 1^2$ which is a circle with centre $(0,-1)$ & Radius 1



$w = \frac{1}{z}$



Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ in z-plane are transformed into region b/w the circle in w-plane.

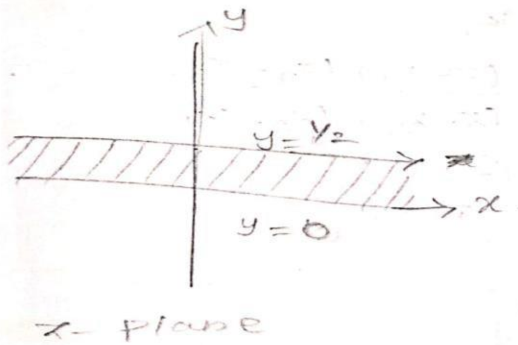
ii) $0 < y < \frac{1}{2}$

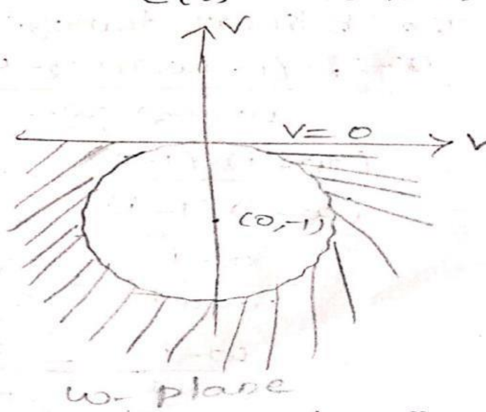
when $y = 0$

$$\frac{-v}{u^2+v^2} = 0$$
$$-v = 0$$
$$v = 0$$

$y = \frac{1}{2}$

$$\frac{-v}{u^2+v^2} = \frac{1}{2}$$
$$u^2+v^2 = -2v$$
$$u^2+v^2+2v+1^2-1^2 = 0$$
$$u^2+(v+1)^2 = 1^2$$ which represents a circle with $C(0,-1)$ & $R(1)$





Hence the infinite strip $0 < y < \frac{1}{2}$ in z-plane are transformed into the region outside the circle $u^2+(v+1)^2 = 1$ and lower half of the w-plane.