Type – II Clairaut's Form $z = px + qy + f(p, q)$

Working Rule:

Complete Integral:

Replace $p \to a$ and $q \to b$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

General Integral:

Put $b = \phi(a)$ in Complete Integral.

1. Solve $z = px + qy + pq$

Soln.:

Given. $z = px + qy + pq \to (1)$

Complete Integral:

$$z = ax + by + ab \to (2) \quad \left[\text{Replace } p \to a,\ q \to b\right.$$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0 \quad \text{and} \quad \frac{\partial z}{\partial b} = 0$$

$$x + b = 0 \qquad y + a = 0$$

$$b = -x \qquad a = -y$$

Subs. a & b in (2)

$$z = -yx - xy - y(-x)$$

$$z = -xy$$

General Integral:

Subs. $b = \phi(a)$ in (2),

$$z = ax + \phi(a)y + a\,\phi(a) \to (3)$$

WKT $\frac{\partial z}{\partial a} = 0$

$$\Rightarrow x + \phi'(a)y + a\phi'(a) + \phi(a) = 0 \to (4)$$

Eliminate 'a' b/w (4) & (3), we get the general Soln.

Q. Solve $z = px + qy + p^2 - q^2$

Soln.: Given. $z = px + qy + p^2 - q^2 \rightarrow (1)$

Complete Integral:

$z = ax + by + a^2 - b^2 \rightarrow (2)$ Replace $p \rightarrow a$, $q \rightarrow b$

Singular Integral:

$\frac{\partial z}{\partial a} = 0$

$x + 2a = 0$

$2a = -x$

$a = \frac{-x}{2}$

$\frac{\partial z}{\partial b} = 0$

$y - 2b = 0$

$y = 2b$

$\Rightarrow b = \frac{y}{2}$

Subs. a & b in (2),

$z = -\frac{x}{2}x + \frac{y}{2}y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$

$= -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$4z = -2x^2 + 2y^2 + x^2 - y^2$

$= -x^2 + y^2$

$y^2 - x^2 = 4z$

General Integral:

Subs. $b = \phi(a)$ in (2)

$z = ax + \phi(a)y - [\phi(a)]^2 + a^2 \rightarrow (3)$

WKT $\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + \phi'(a)y - 2\phi(a)\phi'(a) + 2a = 0 \rightarrow (4)$

Eliminate 'a' b/w (3) & (4), we get the general soln.

3]. Solve $z = px + qy + \sqrt{1+p^2+q^2}$

Soln.:

Given. $z = px + qy + \sqrt{1+p^2+q^2}$

Complete Integral:

$$z = ax + by + \sqrt{1+a^2+b^2} \longrightarrow (A)$$

Singular Integral:

$$\frac{\partial z}{\partial a} = 0$$

$$x + \frac{1\,(2a)}{2\sqrt{1+a^2+b^2}} = 0$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \longrightarrow (1)$$

$$\frac{\partial z}{\partial b} = 0$$

$$y + \frac{1\,(2b)}{2\sqrt{1+a^2+b^2}} = 0$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \longrightarrow (2)$$

Squaring on bothsides.

$$x^2 = \frac{a^2}{1+a^2+b^2} \qquad y^2 = \frac{b^2}{1+a^2+b^2}$$

Now,

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - (x^2+y^2) = 1 - \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - x^2 - y^2 = \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2} = \frac{1}{1+a^2+b^2}$$

Taking square root,

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{1+a^2+b^2}} \Rightarrow \sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$(1) \Rightarrow x = -a\sqrt{1-x^2-y^2} \Rightarrow a = \dfrac{-x}{\sqrt{1-x^2-y^2}}$

$(2) \Rightarrow y = -b\sqrt{1-x^2-y^2} \Rightarrow b = \dfrac{-y}{\sqrt{1-x^2-y^2}}$

$$(A) \Rightarrow z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$= \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$= \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

HW

1]. $z = px + qy + (pq)^{3/2}$

2]. $z = px + qy + p^2q^2$