Fundamentals of the Analysis of Algorithm Efficiency

- Analysis Framework
- Asymptotic Notations and its properties
- Mathematical analysis of Non - Recursive algorithms
- <u>Mathematical analysis of Recursive algorithms</u>

Mathematical analysis of Recursive algorithms

General plan for Analyzing the time efficiency of Recursive algorithm

1. Decide on a parameter (or parameters) indicating an **input's size.**
2. Identify the algorithm's **basic operation.**
3. Check whether the **number of times the basic operation** is executed can vary on different inputs of the same size; if it can, the worst-case, average-case, and best-case efficiencies must be investigated separately.
4. Set up a **recurrence relation**, with an appropriate initial condition, for the number of times the basic operation is executed.
5. Solve the recurrence or, at least, ascertain the **order of growth** of its solution.

Mathematical analysis of Recursive algorithms

- Recursive Function – function that calls itself
- Example 1: Factorial of a given number

$n! = 1 \, . \, . \, . \, . \, . \, (n-1) \, . \, n = (n-1)! * n$ for $n \geq 1$

$F(n) = F(n-1) \, . \, n$ for $n > 0$,

```
ALGORITHM  F(n)
    //Computes n! recursively
    //Input: A nonnegative integer n
    //Output: The value of n!
    if n = 0 return 1
    else return F(n − 1) ∗ n
```

Example 1: Factorial of a given number

- $F(n) = F(n-1) \cdot n$ for $n > 0$
- No.of multiplications ***(Recurrence relation)***

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute} \\ F(n-1)}} + \underbrace{1}_{\substack{\text{to multiply} \\ F(n-1) \text{ by } n}} \quad \text{for } n > 0.$$

- ***Initial condition – sequence***

if n=0 return 1

$n=0$ → no multiplications are done

$$M(0) = 0.$$

the calls stop when $n = 0$

no multiplications when $n = 0$

Example 1: Factorial of a given number

- F(n)=F(n-1) . N
- F(0) = 1
- $M(n) = M(n-1) \quad + 1$

$$= [M(n-2) +1] + 1 = M(n-2) +2$$

$$= [M(n-3)+ 2] + 1 = M(n-3) + 3$$

$M(n) = M(n-i) + i$

If $i=n$,

$M(n) = M(n-n)+n$

$\quad = M(0) + n$

$\quad = n$

Example 2: Towers of Hanoi

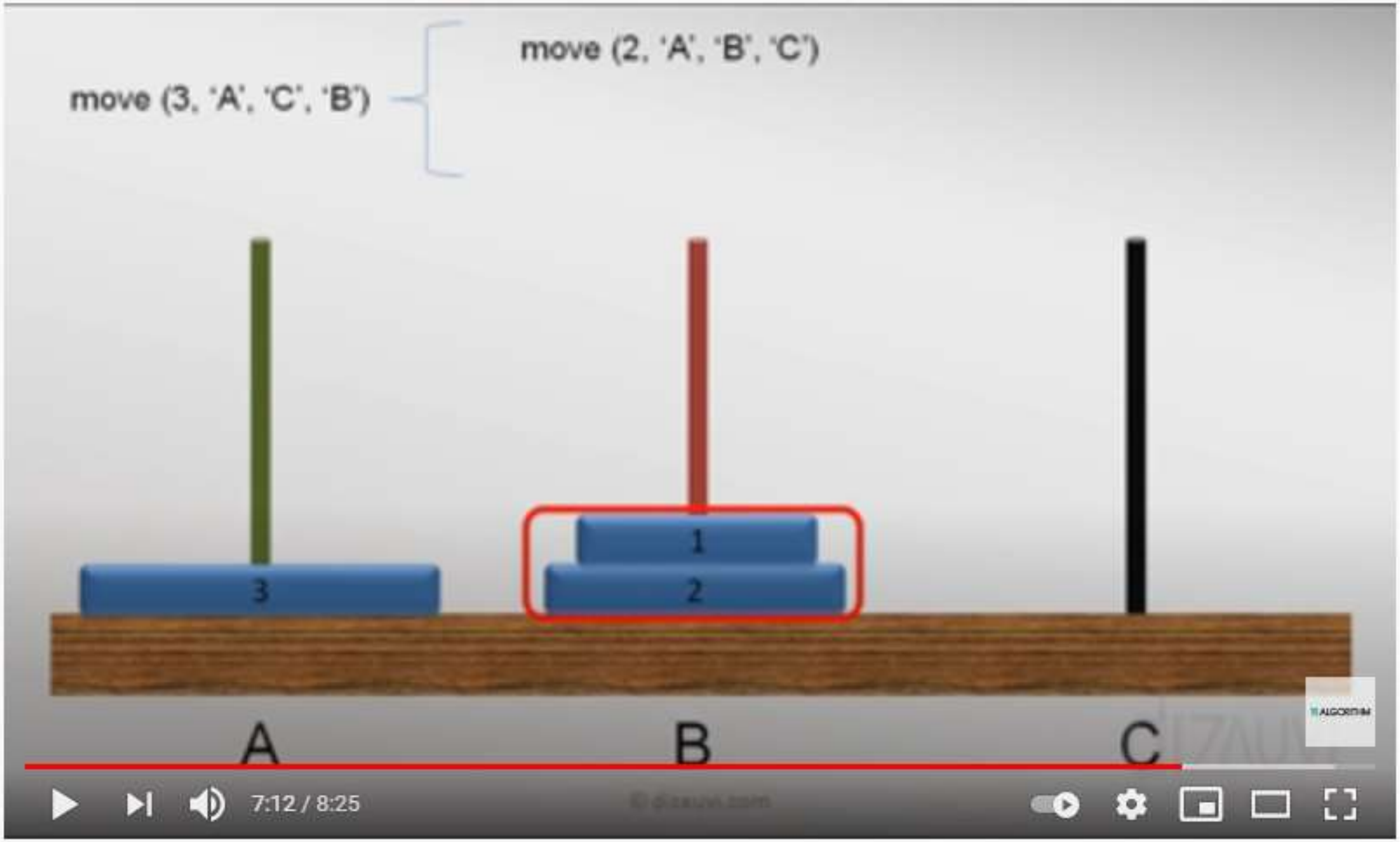
Problem statement : Given n disks of different sizes and 3 rods. Initially all the disks are in the 1st rod, largest on the bottom and smallest on the top.

The goal is to move all the disks to 3rd rod with the help of 2nd rod if essential.

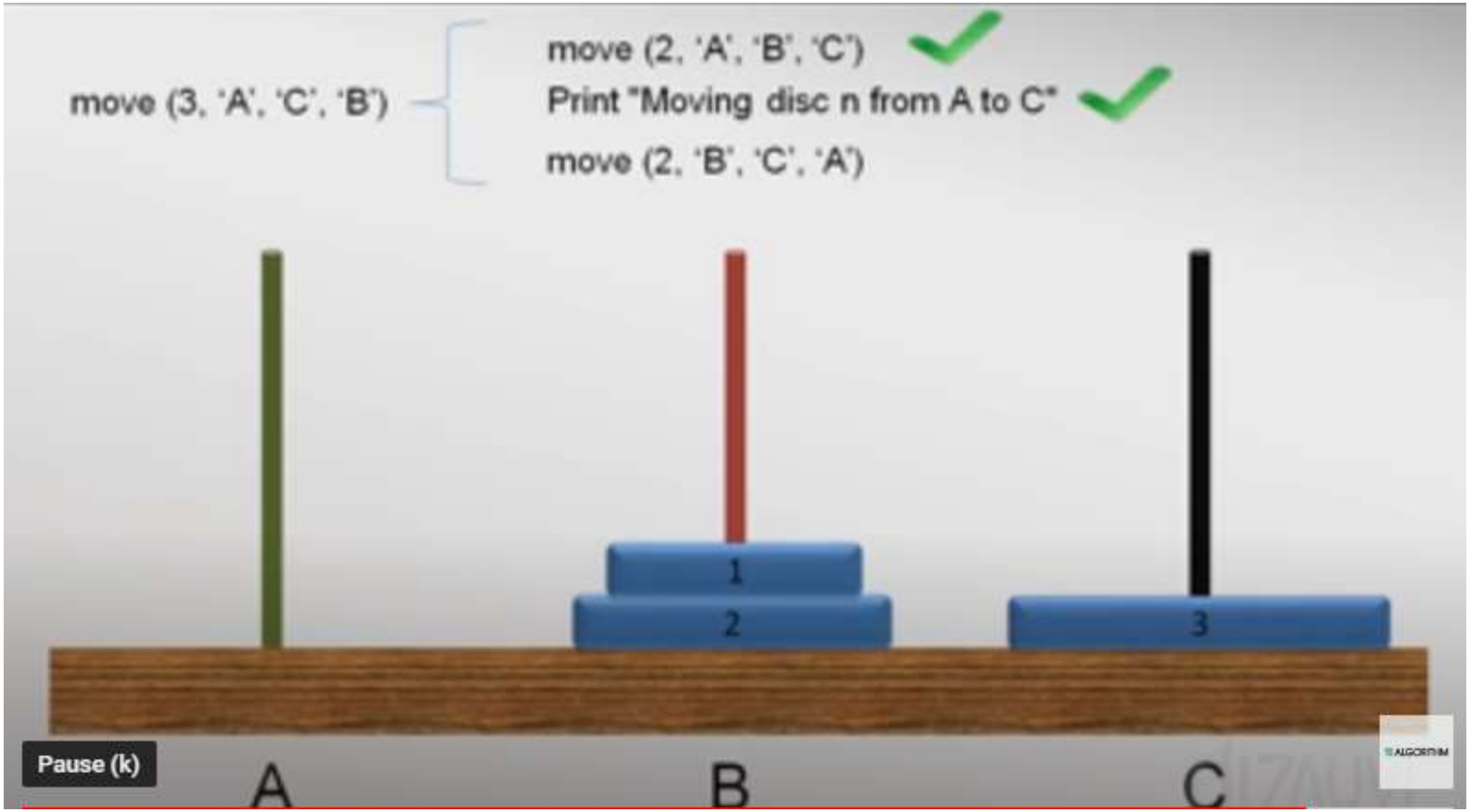
Condition 1: Move one disk at a time

Condition 2: place smaller disk on larger disk

Setting up the Recurrence Relation



Setting up the Recurrence Relation



Example 2: Towers of Hanoi

- Initial condition M(1) = 1

(if there are only one disk we can move to 3rd rod with one move)

- M(n) = M(n-1) + 1 + M(n-1) for n>1. ***Backward Substitution***
- $M(n) = 2M(n-1) + 1$ sub. $M(n-1) = 2M(n-2) + 1$

  $= 2[2M(n-2) + 1] + 1 = 2^2M(n-2) + 2 + 1$ sub. $M(n-2) = 2M(n-3) + 1$

  $= 2^2[2M(n-3) + 1] + 2 + 1 = 2^3M(n-3) + 2^2 + 2 + 1.$

- $2^4M(n-4) + 2^3 + 2^2 + 2 + 1$
- $M(n) = 2^iM(n-i) + 2^{i-1} + 2^{i-2} + \ldots + 2 + 1 = \mathbf{2^iM(n-i) + 2^i - 1.}$
- $[2^4 = 16]\ [2^3+2^2+2^1+1=8+4+2+1 = 15]$
- **Initial condition is n=1, so i= upper bound – lower bound → i=n-1**
- $M(n) = 2^{n-1}M(n-(n-1)) + 2^{n-1} - 1$

  $= 2^{n-1}M(1) + 2^{n-1} - 1 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1.$

Analysis of problems discussed

Problem	Size of the problem	Basic operation	Count of basic operation	Efficiency class
Greatest element in list	n	Comparison inside loop A[i]>maxval	O(n)	Worst /Best
Matrix Muliplication	Order of matrix	Multiplication	O(n^3)	Worst
Element Uniqueness Problem	n	Comparison inside for loop	O(n^2)	Worst
No. of bits in a decimal number	n	Comparison	O($\log_2 n$)	Worst/Best/Avg
Factorial of a given number	n	Multiplication	O(n)	Worst
Towers of hanoi	n	Movements	O(2^n-1)	Worst