Problem 5 Calculate the residue of $f(z)=\dfrac{1-e^{2z}}{z^3}$ at the poles.

Solution:

Given $f(z)=\dfrac{1-e^{2z}}{z^3}$

Here $z=0$ is a pole of order 3

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=0} = \underset{z\to 0}{Lt}\ \frac{1}{2!}\frac{d^2}{dz^2}\left[(z-0)^3\frac{1-e^{2z}}{z^3}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ \frac{d^2}{dz^2}\left[1-e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ \frac{d}{dz}\left[-2e^{2z}\right]$$

$$=\frac{1}{2!}\underset{z\to 0}{Lt}\ -4e^{2z}$$

$$=\frac{1}{2}(-4)=-2.$$

Problem 8 Test for singularity of $\frac{1}{z^2+1}$ and hence find corresponding residues.

Solution:

Let $f(z)=\frac{1}{z^2+1}=\frac{1}{(z+i)(z-i)}$

Here $z=-i$ is a simple pole

$z=i$ is a simple pole

$$Res(z=i)=\underset{z\to i}{Lt}(z-i)\frac{1}{(z+i)(z-i)}$$

$$=\underset{z\to i}{Lt}\frac{1}{(z+i)}=\frac{1}{2i}$$

$$Res(z=-i)=\underset{z\to -i}{Lt}(z+i)\frac{1}{(z+i)(z-i)}=\frac{1}{-2i}.$$