UNIT-4 **COMPLEX INTEGRATION**

PART A (2 Marks)

1. **State Cauchy's integral formula.**

Solution:

If f(z) is analytic inside and on a simple closed curve C and 'a' be any point inside C then $\int_c \frac{f(z)}{z-a}dz = 2\pi i.f(a)$ **where the integration being taken in the positive direction around C.**

2. **Evaluate** $\int_C \frac{z^2+1}{(z-2)(z-3)}dz$ **where c is** $|z|=1$

Solution:

Cauchy's integral formula is $\int_c \frac{f(z)}{z-a}dz = 2\pi i.f(a)$ **[a lies inside C]**

a=2 and 3 lies outside the circle $|z|=1$ **.**

By Cauchy's integral formula

$\int_C \frac{z^2+1}{(z-2)(z-3)}dz$ **=0**

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PART A (2 Marks)

3. **Evaluate** $\int_C \frac{3z^2+7z+1}{(z-3)}dz$ **where c is** $|z|=2$

Solution:

The pole is at z=3 lies outside the circle $|z|=2$ **.**

By Cauchy's integral formula

$\int_C \frac{3z^2+7z+1}{(z-3)}dz$ **=0**

4. **Evaluate** $\int_C \frac{2}{(z-1)(z+3)}dz$ **where c is** $|z-1|=2$

Solution:

$|z-1|=2$ **is a circle whose centre is 1 and radius 2.**

z =-3 lies outside the circle $|z-1|=2$

z=1 lies inside the circle $|z-1|=2$

$$\therefore \int_C \frac{2}{(z-1)(z+3)}dz = \int_C \frac{\left(\frac{2}{z+3}\right)}{(z-1)}dz = 2\pi i f(1) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

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PART A (2 Marks)

5. Evaluate $\int_C (z^2+2z)dz$ **where c is** $|z|=1$

Solution: Let

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}d\theta$$
$$\int_c (z^2+2z)dz = \int_0^{2\pi} [(e^{i\theta})^2 + 2e^{i\theta}]ie^{i\theta}d\theta$$
$$= i\int_0^{2\pi}\left(e^{3i\theta}+2e^{i2\theta}\right)d\theta$$
$$= i\left[\frac{e^{3i\theta}}{3i}+\frac{2e^{i2\theta}}{2i}\right]_0^{2\pi}$$
$$= \left[\frac{e^{3i\theta}}{3}+e^{i2\theta}\right]_0^{2\pi}$$
$$= \left[\frac{\cos 3\theta + i\sin 3\theta}{3}\right]_0^{2\pi} + \left[\cos 2\theta + i\sin 2\theta\right]_0^{2\pi}$$
$$= \frac{1}{3}\left[(1+0)-(1-0)\right]+[(1+0)-(1-0)]$$
$$= 0$$

[or] z^2+2z **is analytic inside and on C.**

$\int_C (z^2+2z)dz$ **=0 by Cauchy's integral formula.**

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PART A (2 Marks)

6. **Evaluate** $\int_C (z-3)^4 dz$ **where c the circle is** $|z-3| = 4$

Solution: Let

$$z - 3 = 4e^{i\theta}$$
$$z = 4e^{i\theta} + 3$$
$$dz = 4ie^{i\theta} d\theta$$
$$\therefore \int_C (z-3)^4 dz = \int_0^{2\pi} \left(4e^{i\theta}\right)^4 4ie^{i\theta} d\theta$$
$$= \int_0^{2\pi} (4)^5 e^{i5\theta} i d\theta$$

$$= \frac{(4)^5}{5} [\cos 5\theta + i \sin 5\theta]_0^{2\pi}$$

7. **Evaluate** $\int_C \frac{dz}{z+4}$ **where c is the circle** $|z| = 2.$

Let $f(z) = 1$ **and** $z = -4$ **lies outside of the circle** $|z| = 2$

$\therefore$ **By cauchy's integral formula, we have** $\int_C \frac{dz}{z+4} = 0$.

8. **Evaluate** $\int_C \frac{e^z dz}{z+1}$ **where c is the circle** $\left|z + \frac{1}{2}\right| = 1.$

Let $f(z) = e^z$ **and** $z = -1$ **lies outside of** $\left|z + \frac{1}{2}\right| = 1.$

By cauchy's integral formula, we have $\int_C \frac{e^z dz}{z+1} = 0$.

9. **Evaluate** $\int_C \frac{dz}{z+3}$ **where c is** $|z| = 1.$

Let $f(z) = 1$ **and** $z = -3$ **lies outside of the circle** $|z| = 1.$

$\therefore$ **By cauchy's integral formula, we have** $\int_C \frac{dz}{z+3} = 0$

10. Evaluate $\int_C \frac{dz}{(z-3)^2}$ **where c is the circle** $|z| = 1.$

$$\int_C \frac{f(z)}{(z-\alpha)^2} = 2\pi i f'(a)$$

a=3 lies outside $|z| = 1.$

$$\int_C \frac{dz}{(z-3)^2} = 0$$

11. Evaluate $\int_C \frac{\cos \pi z}{z-1} dz$ **if c is** $|z| = 2$

$z = 1$ **lies inside the circle** $|z| = 2$

w.k.t cauchy's integral formula is $\int_C \frac{f(z)}{(z-\alpha)} = 2\pi i f(a)$

$$= 2\pi i(-1) \quad \begin{bmatrix} a = 1, & f(z) = \cos \pi z \\ & f(a) = \cos \pi a \\ & f(1) = \cos \pi = 1 \end{bmatrix}$$

$$= -2\pi i$$

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PART A (2 Marks)

12. Using cauchy's integral formula evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$ **where c is** $|z| = \frac{3}{2}$

$z = 1$ **lies inside the circle**

$z = 2$ **lies inside the circle**

$\therefore$ **the given integral can be re-written as** $\int \frac{\frac{\cos \pi z^2}{z-2}}{z-1} dz$

$$f(z) = \frac{\cos \pi z^2}{z-2}$$

$$f(1) = \frac{\cos \pi}{-1} = \frac{-1}{-1} = 1$$

By Cauchy's integral formula $\int \frac{\frac{\cos \pi z^2}{z-2}}{z-1} dz = 2\pi i(1) = 2\pi i$

13. Expand log(1+z) Taylor's series about z=0.

Solution:

$$f(z) = \log(1+z) \Rightarrow f(0) = \log 1 = 0$$
$$f'(z) = \frac{1}{1+z} \Rightarrow f'(0) = 1$$
$$f''(z) = -\frac{1}{(1+z)^2} \Rightarrow f''(0) = -1$$
$$f'''(z) = \frac{2(1+z)}{(1+z)^4} = \frac{2}{(1+z)^3} \Rightarrow f'''(0) = 2$$

The Taylor series of f(z) about the point z=0 is given by

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \frac{z^3}{3!}f'''(0) + ...$$
$$= 0 + \frac{z}{1!} - \frac{z^2}{2!} + 2\frac{z^3}{3!} - ...$$
$$= z - \frac{z^2}{2} + \frac{z^3}{3} - ...$$

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PART A (2 Marks)

14. Expand $\frac{1}{z-2}$ at z=1 in a Taylor's series.

Solution:

$$f(z)=\frac{1}{z-2}\Rightarrow f(1)=-1$$
$$f'(z)=-\frac{1}{(z-2)^2}\Rightarrow f'(1)=-1$$
$$f''(z)=\frac{2(z-2)}{(z-2)^4}=\frac{2}{(z-2)^3}\Rightarrow f''(1)=-2$$

The Taylor series of f(z) about the point z=0 is given by

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots\ldots,\text{here}(a=1)$$
$$=f(1)+\frac{z-1}{1!}f'(1)+\frac{(z-1)^2}{2!}f''(1)+\ldots\ldots$$
$$=-1+(z-1)(-1)+\frac{(z-1)^2}{2}(-2)+\ldots$$
$$=-1-(z-1)-(z-1)^2-\ldots\ldots$$

15. Expand $\frac{z-1}{z+1}$ in Taylor's series about z=1.

Solution:

$$f(z)=\frac{z-1}{z+1}\Rightarrow f(1)=0$$
$$f'(z)=\frac{(z+1)-(z-1)}{(z+1)^2}=\frac{2}{(z+1)^2}\Rightarrow f'(1)=\frac{2}{4}=\frac{1}{2}$$
$$f''(z)=\frac{-2-2(z+1)}{(z+1)^4}=-\frac{4}{(z+1)^3}\Rightarrow f''(1)=-\frac{4}{8}=-\frac{1}{2}$$

So the Taylor's series for $\frac{z-1}{z+1}$ about z=1 is

$$f(z)=f(a)+\frac{z-a}{1!}f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots\ldots,\text{here}(a=1)$$

$$=f(1)+\frac{z-1}{1!}f'(1)+\frac{(z-1)^2}{2!}f''(1)+\ldots\ldots$$

$$=0+\left(\frac{1}{2}\right)(z-1)+\left(\frac{1}{2}\right)\frac{(z-1)^2}{2}+\ldots$$

16. Find Laurent's series of $f(z)=z^2e^{\frac{1}{z}}$ about z=0.

Solution:

Clearly f(z) is not analytic at z=0.

$$f(z)=z^2e^{\frac{1}{z}}$$

$$=z^2\left[1+\frac{\left(\frac{1}{z}\right)}{1!}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\ldots\right]$$

$$=z^2+\frac{z}{1!}+\frac{1}{2!}+\ldots$$

17. Obtain the Laurent expansion of the function $\frac{e^z}{(z-1)^2}$ in the neighbourhood of its singular point. Hence find the residue at the point.

Solution:

Here z=1 is a singular point.

$$f(z)=\frac{e^z}{(z-1)^2}=\frac{e^{z+1-1}}{(z-1)^2}=\frac{e^{z+1}.e^1}{(z-1)^2}$$

$$=\frac{e^1}{(z-1)^2}\left[1+\frac{(z-1)}{1!}+\frac{(z-1)^2}{2!}+\ldots\ldots\right]$$

$$=e^1\left[\frac{1}{(z-1)^2}+\frac{1}{(z-1)}+\frac{1}{2}+\ldots\ldots\right]$$

UNIT-4 **COMPLEX INTEGRATION**

PART A (2 Marks)

Residue at the point z=1 is the coefficient of $\frac{1}{z-1}$

$$\therefore \text{Res}\{f(z),1\} = e$$

18. State Cauchy's Residue theorem.
If f(z) be analytic at all points inside and on a simple closed curve C,except

for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ **inside C,then**

$$\int_c f(z)dz = 2\pi i[\text{sum of the residues of } f(z) \text{ at } z_1, z_2, \ldots, z_n]$$

$$= 2\pi i \sum_{i=1}^{n} R_i \text{ where } R_i \text{ is the residue of } f(z) \text{ at } z = z_i$$

19. Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ **at the pole**

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$

Z=-1 is a pole of order 2

$$\text{Res}\{f(z)\}_{z=-1} = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{e^{2z}}{(z+1)^2}\right]$$

$$= \lim_{z\to -1} \frac{d}{dz}\left[e^{2z}\right]$$

$$= \lim_{z\to -1} 2e^{2z}$$

$$= 2e^{-2}$$

20. Find the residue of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ **at z=-2.**

Z=1 is a pole of order 2

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PART A (2 Marks)

Z=-2 is a pole of order 1

Res $\{f(z)\}_{z=-2} = \lim_{z \to -2}\left[(z+2) \dfrac{z^2}{(z-1)^2(z+2)}\right]$

$$= \lim_{z \to -2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9} = \frac{2}{3}$$

21. Find the residue of $f(z) = \dfrac{z+1}{z(z+2)}$

Z=0 is a pole of order 1

Z=2 is a pole of order 1

$$\text{Res}\{f(z)\}_{z=0} = \lim_{z \to 0} z \frac{z+1}{z(z-2)}$$

$$\text{Res}\{f(z)\}_{z=2} = \lim_{z \to 2}(z-2)\frac{z+1}{z(z-2)} = \frac{-3}{2}$$

Z=2 is a simple pole

$$\text{Res}\{f(z)\}_{z=2} = \lim_{z \to 2}(z-2)\frac{4}{z^3(z-2)} = \frac{1}{2}$$

22. Obtain the residues of the function $f(z) = \dfrac{z-3}{(z+1)(z-2)}$ **at its poles.**

Z=-1 is a pole of order 1

Z=-2 is a pole of order 1

$$\text{Res}\{f(z)\}_{z=-1} = \lim_{z \to -1}(z+1)\frac{(z-3)}{(z+1)(z+2)} = -4$$

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PART A (2 Marks)

$$\operatorname{Res}\{f(z)\}_{z=-2} = \lim_{z\to -2}(z+2)\frac{(z-3)}{(z+1)(z+2)} = -5$$

23. Consider the function $f(z)=\frac{\sin z}{z^4}$ .Find the pole and its order.

$$f(z)=\frac{\sin z}{z^4}=\frac{1}{z^4}\left[z-\frac{z^3}{3}+\frac{z^5}{5}-\ldots\ldots\ldots\right]$$

$$=\frac{1}{z^3}\left[1-\frac{z^2}{3}+\frac{z^4}{5}-\ldots\ldots\ldots.\right]$$

Z=0 is a pole of order 3

24. Define pole and simple poles.

A point z=a is said to be a pole f(z) of order n if we can find a positive integer n such that $\lim_{z\to a}(z-a)^n f(z)\neq 0$

A pole of order one is called a simple pole.

Example :

$$f(z)=\frac{1}{(z-4)^2(z-3)^3(z-1)}$$

Hence z=1 is a simple pole of order 1

Z=4 is a simple pole of order 2

Z=3 is a simple pole of order 3.

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PART A (2 Marks)

25. Find the regularities of the function $f(z)=\frac{\cot \pi z}{(z-a)^3}$

Given $f(z)=\frac{\cot \pi z}{(z-a)^3}=\frac{\cos \pi z}{\sin \pi z (z-a)^3}$

Singular points are poles and are given by Dr=0.

$$\sin \pi z = 0, \quad (z-a)^3 = 0 \Rightarrow z = a \text{ is a singular pole of order } 3$$
$$\sin \pi z = \sin n \sin n\pi \quad \text{where } n = 0, \pm1, \pm2, \ldots\ldots$$
$$\pi z = n\pi \Rightarrow n = \pi$$
$$z = n = 0, \pm1, \pm2, \ldots\ldots \text{ are simple poles.}$$

26. Find the principal part and residue at the pole of $f(z)=\frac{2z+3}{(z+2)^2}=(2z+3)(z+2)^{-2}$

[since principal part is negative powers]

$$(z+2)^{-2} \Rightarrow z=-2 \text{ is a singular pole of order } 2$$

$$\operatorname{Res}\{f(z)\}_{z=-2} = \lim_{z \to -2} \frac{d}{dz}\left[(z+2)^2 . \frac{(2z+3)}{(z+2)^2}\right] = 2$$

27. Evaluate $\int_c \frac{dz}{z-2}$ **where C is the square with vertices (0,0),(1,0),(1,1) and (0,1).**

Given $\int_c \frac{dz}{z-2}$

$$\text{Here } f(z) = 1$$
$$a = 2 \text{ lies outside}$$
$$\therefore \text{By Cauchy's integral formula } \int_c \frac{dz}{z-2} = 0.$$