(1) Electromagnetic wave :

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Wave equations for plane electromagnetic wave in free space

Wave equation in terms of $\vec{E}$ in free space: $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\delta^2 \vec{E}}{\delta t^2}$

Wave equation in terms of $\vec{B}$ in free space: $\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\delta^2 \vec{B}}{\delta t^2}$.

Maxwell's equation for electromagnetic field is

$$\vec{\nabla} \cdot \vec{D} = \rho \text{ ——— (i)} \qquad \vec{\nabla} \cdot \vec{B} = 0 \text{ ——— (ii)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} \text{ ——— (iii)} \qquad \vec{\nabla} \times \vec{H} = \vec{J} + \frac{\delta \vec{D}}{\delta t} \text{ ——— (iv)}$$

For free space (or perfect dielectric medium), $\rho = 0$ & $\vec{J} = 0$

Hence Maxwell's equation for electromagnetic field in free space will become

$$\vec{\nabla} \cdot \vec{D} = 0 \text{ ——— (a)} \qquad \vec{\nabla} \cdot \vec{B} = 0 \text{ ——— (b)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\delta \vec{B}}{\delta t} \text{ ——— (c)} \qquad \vec{\nabla} \times \vec{H} = \frac{\delta \vec{D}}{\delta t} \text{ ——— (d)}$$

Derivation of wave equation, $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\delta^2 \vec{E}}{\delta t^2}$ in free space

From equⁿ (c), $\vec{\nabla} \times \vec{E} = -\frac{\delta \vec{B}}{\delta t}$

Taking curl both sides, we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\delta \vec{B}}{\delta t}\right)$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \text{ ——— (1)}$$

Now $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{E}(\vec{\nabla} \cdot \vec{\nabla})$

$$\left[\because \vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})\right]$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}\left(\vec{\nabla} \cdot \frac{\vec{D}}{\varepsilon_0}\right) - \vec{E}(\nabla^2) \qquad \because \vec{E} = \frac{\vec{D}}{\varepsilon_0} \text{ and } \vec{\nabla} \cdot \vec{\nabla} = \nabla^2$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{\varepsilon_0} \vec{\nabla}(0) - \nabla^2 \vec{E} \qquad \because \vec{\nabla} \cdot \vec{D} = 0 \text{ from (a)}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\nabla^2 \vec{E} \text{ ——— (2)}$$

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Now $\frac{\delta}{\delta t}(\vec{\nabla}\times\vec{B}) = \frac{\delta}{\delta t}(\vec{\nabla}\times\mu_0\vec{H})$ $\quad\because \vec{B} = \mu_0\vec{H}$

$$= \mu_0 \cdot \frac{\delta}{\delta t}(\vec{\nabla}\times\vec{H})$$

$$= \mu_0 \cdot \frac{\delta}{\delta t}\left(\frac{\delta\vec{D}}{\delta t}\right) \quad \because \vec{\nabla}\times\vec{H} = \frac{\delta\vec{D}}{\delta t} \text{ from (d)}$$

$$= \mu_0 \frac{\delta^2}{\delta t^2}(\varepsilon_0\vec{E}) \quad \because \vec{D} = \varepsilon_0\vec{E}$$

$$\frac{\delta}{\delta t}(\vec{\nabla}\times\vec{B}) = \mu_0\varepsilon_0\frac{\delta^2\vec{E}}{\delta t^2} \quad \text{———— (3)}$$

Using equⁿˢ (2) and (3) in equⁿ (1), we get

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\delta^2\vec{E}}{\delta t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\delta^2\vec{E}}{\delta t^2}}$$

It is equation of wave in terms of $\vec{E}$ in free space.

In component form,

$$\frac{\delta^2E_x}{\delta x^2} = \mu_0\varepsilon_0\frac{\delta^2E_x}{\delta t^2}$$

$$\frac{\delta^2E_y}{\delta y^2} = \mu_0\varepsilon_0\frac{\delta^2E_y}{\delta t^2}$$

$$\frac{\delta^2E_z}{\delta z^2} = \mu_0\varepsilon_0\frac{\delta^2E_z}{\delta t^2}$$

These are equation of waves in free space in terms of three mutually perpendicular components of $\vec{E}$.

<u>Derivation of wave equation $\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\delta^2\vec{B}}{\delta t^2}$ in free space</u>

From equⁿ (d) $\quad \vec{\nabla}\times\vec{H} = \frac{\delta\vec{D}}{\delta t}$

Taking curl both sides, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \frac{\delta}{\delta t}(\vec{\nabla}\times\vec{D}) \quad \text{———— (4)}$$

Now $\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \vec{H}(\vec{\nabla}\cdot\vec{\nabla})$

$[\because \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})]$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{B}}{\mu_0}\right) - \frac{\vec{B}}{\mu_0}(\nabla^2) \quad \because \vec{H} = \frac{\vec{B}}{\mu_0}, \vec{\nabla}\cdot\vec{\nabla} = \nabla^2$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \frac{1}{\mu_0}\vec{\nabla}(0) - \frac{1}{\mu_0}\nabla^2\vec{B} \quad \because \vec{\nabla}\cdot\vec{B}=0 \text{ from (b)}$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = -\frac{1}{\mu_0}\cdot\nabla^2\vec{B} \quad \text{——— (5)}$$

Now $\frac{\delta}{\delta t}(\vec{\nabla}\times\vec{D}) = \frac{\delta}{\delta t}(\vec{\nabla}\times\varepsilon_0\vec{E}) = \varepsilon_0\frac{\delta}{\delta t}(\vec{\nabla}\times\vec{E}) \quad \because \vec{D}=\varepsilon_0\vec{E}$

$$\Rightarrow \frac{\delta}{\delta t}(\vec{\nabla}\times\vec{D}) = \varepsilon_0\frac{\delta}{\delta t}\left(-\frac{\delta\vec{B}}{\delta t}\right) \quad \because \vec{\nabla}\times\vec{E} = -\frac{\delta\vec{B}}{\delta t} \text{ from (c)}$$

$$\Rightarrow \frac{\delta}{\delta t}(\vec{\nabla}\times\vec{D}) = -\varepsilon_0\frac{\delta^2\vec{B}}{\delta t^2} \quad \text{——— (6)}$$

Using equ^ns (5) and (6) in equ^n (4), we get

$$-\frac{1}{\mu_0}\nabla^2\vec{B} = -\varepsilon_0\frac{\delta^2\vec{B}}{\delta t^2}$$

$$\Rightarrow \boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\delta^2\vec{B}}{\delta t^2}}$$

It is equation of wave in terms of $\vec{B}$ in free space.

In component form,

$$\frac{\delta^2 B_x}{\delta x^2} = \mu_0\varepsilon_0\frac{\delta^2 B_x}{\delta t^2}$$

$$\frac{\delta^2 B_y}{\delta y^2} = \mu_0\varepsilon_0\frac{\delta^2 B_y}{\delta t^2}$$

$$\frac{\delta^2 B_z}{\delta z^2} = \mu_0\varepsilon_0\frac{\delta^2 B_z}{\delta t^2}$$

These are equation of waves in free space in terms of three mutually perpendicular components of $\vec{B}$.

Determination of velocity of electromagnetic wave

Equation of electromagnetic wave in terms of $\vec{E}$ in free space is

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\delta^2\vec{E}}{\delta t^2} \quad \text{——— (1)}$$

Wave equation in general form is

$$\nabla^2\psi = \frac{1}{v^2}\cdot\frac{\delta^2\psi}{\delta t^2} \quad \text{——— (2)}$$

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Comparing equⁿˢ (1) and (2) we get

$$\frac{1}{v^2} = \mu_0 \varepsilon_0 \Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C = \text{velocity of light in free space.}$$

where $\mu_0 = 4\pi \times 10^{-7}\ Tm/A$ and $\varepsilon_0 = \frac{1}{36\pi \times 10^{9}}\ C^2/Nm^2$

velocity of electromagnetic wave (light) in free space i.e

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times \frac{1}{36\pi \times 10^{9}}}} = \sqrt{9 \times 10^{16}}$$

$$c = 3 \times 10^{8}\ m/s.$$

In a medium, velocity of electromagnetic wave will be

$$\boxed{v = \frac{1}{\sqrt{\mu \varepsilon}}}$$

Ques:- Derive (or describe) the equation for electromagnetic wave in free space using Maxwell's field equation.

Ques:- Short notes on velocity of electromagnetic wave.