Chapter 19: Plane Wave Propagation in Free Space

Chapter Learning Objectives: After completing this chapter the student will be able to:

- Distinguish between waves moving in the positive direction and those moving in the negative direction, and prove that both are a solution to the wave equation.
- Explain what a plane wave is and under what condition it is a solution to the wave equation.
- Calculate the phase velocity, wave number, radial frequency, frequency, period, wavelength, and direction of travel for a plane wave.
- Calculate the electric or magnetic field (given the other) for a plane wave propagating through free space.
- Determine the magnitude and direction of energy flow for a plane wave.

You can watch the video associated with this chapter at the following link:

Historical Perspective: Guglielmo Marconi (1874-1937) was an Italian inventor and electrical engineer. He spent his life studying radio wave transmissions, and he is recognized as the inventor of radio. He shared the 1909 Nobel Prize in Physics for the development of “wireless telegraphy” (radio).

Photo credit: https://upload.wikimedia.org/wikipedia/commons/0/0d/Guglielmo_Marconi.jpg, [Public domain], via Wikimedia Commons.

19.1 General Solution to the Wave Equation

Recall that we ended chapter 18 with the derivation of the one-dimensional scalar wave equation, which was derived from Maxwell's equations:

$$\frac{\partial^2 E_y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = 0 \qquad \text{(Copy of Equation 18.42)}$$

Our focus in this chapter will be solving this equation and investigating the behavior of those solutions. The most general form of solution to the wave equation is a sum (superposition) of two waves—one moving to the left and one moving to the right:

$$E_y(z,t) = F(z-ct) + G(z+ct) \qquad \text{(Equation 19.1)}$$

Let's first consider the F(z-ct) term. When we say that a wave is "moving to the right," that means that any point on the wave front (say, a peak or a valley) will be moving to the right. That peak will always correspond to the same value of z-ct, which means that z-ct will be a constant:

$$z - ct = constant \qquad \text{(Equation 19.2)}$$

Taking the derivative of both sides of this equation, we find:

$$\frac{dz}{dt} - c\frac{dt}{dt} = \frac{d(constant)}{dt} \qquad \text{(Equation 19.3)}$$

We know that dt/dt=1, and the derivative of a constant is zero, so:

$$\frac{dz}{dt} - c = 0 \qquad \text{(Equation 19.4)}$$

Moving the c to the right side, we find:

$$\frac{dz}{dt} = c \qquad \text{(Equation 19.5)}$$

This proves that the F(z-ct) portion of the wave is moving toward the right (since the velocity is positive), with a velocity of c. A similar derivation can be used to prove that G(z+ct) is moving toward the left with a velocity of –c.

Let's prove that Equation 19.1 is a solution to the wave equation. We will need to calculate the second derivative of $E_y(z,t)$ with respect to both t and z, and then we can show that this function satisfies the wave equation. We will first define two helper equations:

$$\alpha = z - ct \qquad \text{(Equation 19.6)}$$

$$\beta = z + ct \qquad \text{(Equation 19.7)}$$

Substituting these into Equation 19.1, we find:

$$E_y(z,t) = F(\alpha) + G(\beta) \qquad \text{(Equation 19.8)}$$

The first derivative of this function with respect to t can be found by applying the chain rule:

$$\frac{\partial E_y}{\partial t} = \frac{\partial F(\alpha)}{\partial t} + \frac{\partial G(\beta)}{\partial t} \qquad \text{(Equation 19.9)}$$

$$\frac{\partial E_y}{\partial t} = \frac{\partial F}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial t} + \frac{\partial G}{\partial \beta} \cdot \frac{\partial \beta}{\partial t} \qquad \text{(Equation 19.10)}$$

We can use Equations 19.6 and 19.7 to evaluate the derivatives of a and b with respect to t:

$$\frac{\partial E_y}{\partial t} = \frac{\partial F}{\partial \alpha} \cdot (-c) + \frac{\partial G}{\partial \beta} \cdot (c) \qquad \text{(Equation 19.11)}$$

A similar process is used to evaluate the second derivative if E_y with respect to t:

$$\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 F}{\partial \alpha^2} \cdot (-c)^2 + \frac{\partial^2 G}{\partial \beta^2} \cdot (c)^2 \qquad \text{(Equation 19.12)}$$

The same procedure is used to evaluate the second derivative if E_y with respect to z:

$$\frac{\partial E_y}{\partial z} = \frac{\partial F(\alpha)}{\partial z} + \frac{\partial G(\beta)}{\partial z} \qquad \text{(Equation 19.13)}$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial F}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial z} + \frac{\partial G}{\partial \beta} \cdot \frac{\partial \beta}{\partial z} \qquad \text{(Equation 19.14)}$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial F}{\partial \alpha} \cdot (1) + \frac{\partial G}{\partial \beta} \cdot (1) \qquad \text{(Equation 19.15)}$$

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2 F}{\partial \alpha^2} \cdot (1)^2 + \frac{\partial^2 G}{\partial \beta^2} \cdot (1)^2 \qquad \text{(Equation 19.16)}$$

We will now substitute Equations 19.12 and 19.16 into the left side of the wave equation (Equation 18.42):

$$\frac{\partial^2 E_y}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 F}{\partial \alpha^2} + \frac{\partial^2 G}{\partial \beta^2} - \frac{1}{c^2}\left[\frac{\partial^2 F}{\partial \alpha^2} \cdot c^2 + \frac{\partial^2 G}{\partial \beta^2} \cdot c^2\right] \qquad \text{(Equation 19.17)}$$

All the terms on the right side of this equation cancel each other out, leaving the wave equation:

$$\frac{\partial^2 E_y}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 E_y}{\partial z^2} = 0 \qquad \text{(Equation 19.18)}$$

Thus, we have proven that a combination of waves moving toward the left and right (or either one alone) is a solution to the wave equation. If we can figure out how to get an electromagnetic wave moving at a constant velocity leftward or rightward, it should continue in that direction forever at a constant velocity c. This is actually a very profound result, and it is far more general than it appears at first glance. We will spend a lot of the rest of our time talking about harmonic (sinusoidal) waves, but this result shows that any wave (square, rectangular, or arbitrary) will propagate equally well in free space.

Example 19.1: Which of the following functions is a solution to the one-dimensional scalar wave equation?

a.) $E_y(z,t) = 4z - 4ct + 5z + 5ct$

b.) $E_y(z,t) = 4z - 4ct + 5z + 10ct$

c.) $E_y(z,t) = sin(k(z - ct))$

d.) $E_y(z,t) = 100e^{-(z-ct)^2}$

e.) $E_y(z,t) = (z - ct)(z + ct)$

19.2 Plane Waves as a Solution to the Wave Equation

A "plane wave" seems like a really complicated subject, but it's actually one of the simplest types of waves there are. A plane wave only has dependence (and, therefore, propagation) in one direction. In the figure below, the plane wave is propagating in the z direction. Notice that the polarization is in the y direction (thus, it is $E_y(z)$), but the wave is propagating only in the z direction.

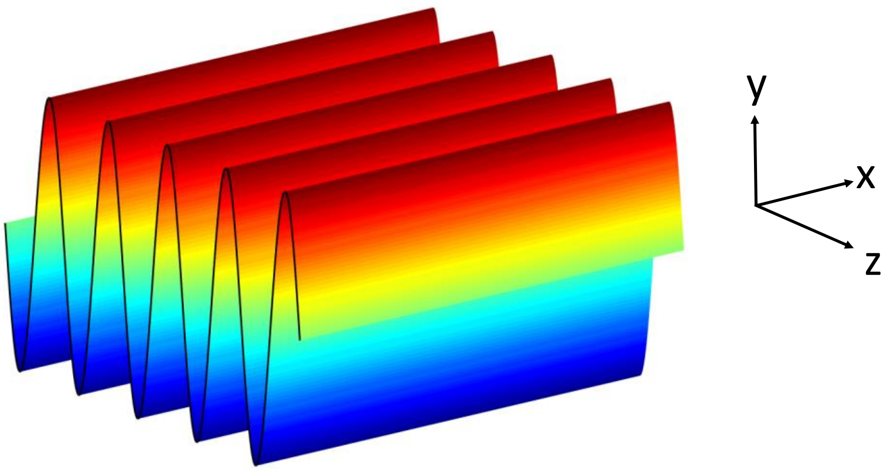

Figure 19.1. A plane wave $E_y(z)$ has no x or y dependence.

We define the **wavelength λ** to be the distance between successive peaks. Since the wave has no dependence in the x or y direction, we can simplify the description of the wave as follows:

$$E_y(x, y, z, t) = E_y(z, t) \qquad \text{(Equation 19.19)}$$

If we also know that the wave is time harmonic, then the z and t dependence can be separated as follows:

$$E_y(z, t) = E_y(z)e^{j\omega t} \qquad \text{(Equation 19.20)}$$

Let's investigate whether (and how) this function will satisfy the wave equation. First, we'll determine the second derivative of E_y with respect to z:

$$\frac{\partial^2 E_y(z, t)}{\partial z^2} = \frac{\partial^2 E_y(z)}{\partial z^2} e^{j\omega t} \qquad \text{(Equation 19.21)}$$

Next, we'll calculate the second derivative with respect to t:

$$\frac{\partial^2 E_y(z, t)}{\partial t^2} = E_y(z)(j\omega)^2 e^{j\omega t} \qquad \text{(Equation 19.22)}$$

Substituting Equations 19.21 and 19.22 into the wave equation, we find:

$$\frac{\partial^2 E_y(z)}{\partial z^2} e^{j\omega t} - \frac{1}{c^2} E_y(z)(j\omega)^2 e^{j\omega t} = 0 \quad \text{(Equation 19.23)}$$

This can be simplified as follows:

$$\frac{\partial^2 E_y(z)}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 E_y(z) = 0 \quad \text{(Equation 19.24)}$$

If we define k (called the **wave number**) as follows:

$$\boxed{k = \frac{\omega}{c}} \quad \text{(Equation 19.25)}$$

Then Equation 19.24 simplifies to:

$$\frac{\partial^2 E_y(z)}{\partial z^2} + k^2 E_y(z) = 0 \quad \text{(Equation 19.26)}$$

This is known as the **one-dimensional Helmholtz equation**, and it is a well-known differential equation with a well-known solution:

$$E_y(z) = ae^{-jkz} + be^{jkz} \quad \text{(Equation 19.27)}$$

Adding the time dependence back into this result, we obtain:

$$E_y(z,t) = ae^{-jkz}e^{j\omega t} + be^{jkz}e^{j\omega t} \quad \text{(Equation 19.28)}$$

Which can be written as:

$$E_y(z,t) = ae^{j(\omega t - kz)} + be^{j(\omega t + kz)} \quad \text{(Equation 19.29)}$$

Notice that this function has the form of Equation 19.1 as long as Equation 19.25 is followed, so it is guaranteed to be a solution to the wave equation. In order to ensure that Equation 19.29 is a solution to the wave equation, the following relationship must be true.

$$v_\phi = \frac{dz}{dt} = \pm c = \pm \frac{\omega}{k} \quad \text{(Equation 19.30)}$$

The positive signs correspond to a right-moving wave, and the negative signs to a left-moving wave.

We now have several variables that describe the time and spatial variation of E(z,t): wavelength λ, wave number k, radial frequency ω, frequency f, period T, and phase velocity v_ϕ. Figure 19.2 is designed to graphically demonstrate the relationships among these six variables. If you know two of the variables, you will be able to calculate the other four.

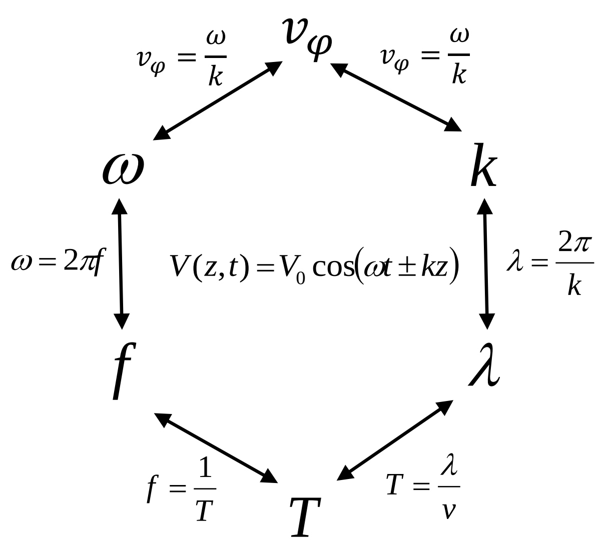

Figure 19.2. Relationships among v_ϕ, k, λ, T, f, and ω.

Example 19.2: An electric field in a vacuum is determined to be polarized in the y direction. It has a peak of 100μV/m at z=0, and the next peak is at z=5cm. Determine v_ϕ, k, λ, T, f, and ω for this wave and write a complete function for $E_y(z,t)$

19.3 Characteristic Impedance

We can derive a very useful relationship among the electric field, magnetic field, and the power associated with a wave. We will start with a known electric field:

$$\boxed{\mathbf{E}(z,t) = ae^{j(\omega t - kz)}\mathbf{a_y}} \quad \text{(Equation 19.31)}$$

Notice that, although we have omitted the y subscript from E(z,t), we know that this wave is polarized in the y direction because of the unit vector. We also know that the wave is moving in the positive z direction because the exponent has an ωt-kz term.

Next, we will write Faraday's Law for time-harmonic plane waves:

$$\nabla \times \mathbf{E}(z,t) = -j\omega\mathbf{B}(z,t) = -j\omega\mu_0\mathbf{H}(z,t) \quad \text{(Equation 19.32)}$$

Solving this equation for **H**(z,t), we find:

$$\mathbf{H}(z,t) = -\frac{1}{j\omega\mu_0}\nabla \times \mathbf{E}(z,t) \quad \text{(Equation 19.33)}$$

Recalling from Equation 11.3 that the curl of a function can be calculated from a determinant:

$$\mathbf{H}(z,t) = -\frac{1}{j\omega\mu_0}\begin{vmatrix} \mathbf{a_x} & \mathbf{a_x} & \mathbf{a_x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & ae^{j(\omega t - kz)} & 0 \end{vmatrix} \quad \text{(Equation 19.34)}$$

Then solving that equation using the method of diagonals, we obtain:

$$\mathbf{H}(z,t) = -\frac{1}{j\omega\mu_0}\frac{\partial(ae^{j(\omega t - kz)})}{\partial z}(-\mathbf{a_x}) \quad \text{(Equation 19.35)}$$

Performing the partial derivative, we find:

$$\mathbf{H}(z,t) = -\frac{1}{j\omega\mu_0}(-jk)ae^{j(\omega t - kz)}(-\mathbf{a_x}) \quad \text{(Equation 19.36)}$$

Simplifying this expression, we find:

$$\boxed{\mathbf{H}(z,t) = -\frac{k}{\omega\mu_0}ae^{j(\omega t - kz)}(\mathbf{a_x})} \quad \text{(Equation 19.37)}$$

Recall that we already determined that the direction of wave propagation (which we will now call $\mathbf{a_k}$) was in the positive z direction:

$$\mathbf{a_k} = \mathbf{a_z} \quad \text{(Equation 19.38)}$$

Now we have found that the electric field is polarized in the $+\mathbf{a_y}$ direction, and the magnetic field is polarized in the $-\mathbf{a_x}$ direction. As Figure 19.3 illustrates, we can then find the direction of the Poynting vector, which is always in the direction of **E** x **H**.

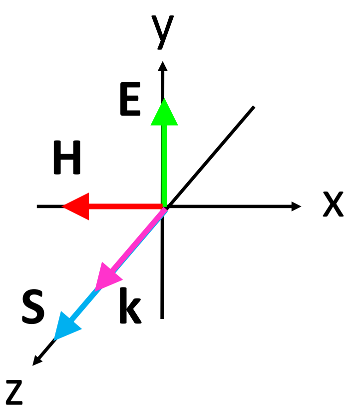

Figure 19.3. Finding the direction of S.

When we take the cross product of **E** with **H**, we find that $\mathbf{a_S}$ is pointing in the $+\mathbf{a_z}$ direction. Thus, we see that the direction of propagation is also the direction of energy transfer:

$$\mathbf{a_S} = \mathbf{a_k} \quad \text{(Equation 19.39)}$$

This is a universal and highly useful result, which also makes good intuitive sense. Of course, the wave will carry energy in the direction it is propagating. But it is always reassuring when a result derived from our formal mechanisms is intuitive in retrospect.

Now we have a good understanding of the relationships among the directions of **E**, **H**, **S**, and **k**. Let's consider their magnitudes. One interesting observation is that the ratio of E divided by H has units of Ω:

$$\frac{E}{H} = \frac{V/m}{A/m} = \Omega \quad \text{(Equation 19.40)}$$

We will call this ratio the **characteristic impedance**. It can be calculated as follows:

$$Z_c = \left|\frac{\mathbf{E}(z,t)}{\mathbf{H}(z,t)}\right| = \frac{ae^{j(\omega t - kz)}}{\frac{k}{\omega\mu_0}ae^{j(\omega t - kz)}} = \frac{\omega\mu_0}{k} \quad \text{(Equation 19.41)}$$

While Equation 19.41 is completely valid, we can rearrange it to make it more useful:

$$Z_c = \frac{\omega\mu_0}{k} = \frac{kc\mu_0}{k} = c\mu_0 = \frac{\mu_0}{\sqrt{\mu_o\epsilon_0}} = \sqrt{\frac{\mu_o}{\epsilon_0}} \quad \text{(Equation 19.42)}$$

Thus, the characteristic impedance (the ratio of electric field magnitude divided by magnetic field magnitude) is only dependent on two universal constants, μ_0 and ε_0. Let's calculate the numerical value of this constant:

$$\boxed{Z_0 = \sqrt{\frac{\mu_o}{\epsilon_0}} = \sqrt{\frac{1.2566 \times 10^{-6}}{8.854 \times 10^{-12}}} = 377\Omega} \quad \text{(Equation 19.43)}$$

Thus, in free space (aka, a vacuum), the magnitude of the electric field will always be 377 times larger than the magnitude of the magnetic field, when using standard SI units.

We can combine our new understanding of the relationships between the direction and magnitude of electric fields and magnetic fields by writing an equation that allows us to find H given E and the direction of propagation:

$$\boxed{H(z,t) = \frac{1}{Z_c}(\mathbf{a_k} \times \mathbf{E}(z,t))} \quad \text{(Equation 19.44)}$$

Example 19.3: If an electric field in a vacuum is known to obey the following equation, what is the direction of wave propagation? Determine a complete expression for the magnetic field that corresponds to this electric field.

$$\mathbf{E}(z,t) = 100cos(2\pi(10^9 t - 40z))\mathbf{a_y} mV/m$$

So, we now understand that both the electric field and the magnetic field vary sinusoidally in both space and time. They are perpendicular to each other, and the direction of propagation is perpendicular to both of them. We refer to this combination as an electromagnetic wave, because neither wave would persist without the other, and propagation is only possible when both waves are present, at the same frequency, and in phase with one another. Figure 19.4 illustrates the relationship among the electric field, the magnetic field, and wave propagation.

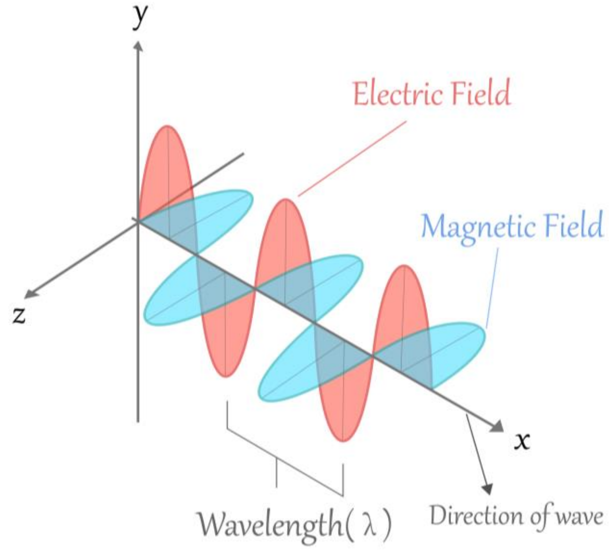

Figure 19.4. Propagation of an electromagnetic wave.
(Image credit: Bigstock.com image #179446048, used with permission.)

19.4 Poynting Vectors and Plane Waves

In Figure 19.3, we saw how the Poynting vector always points in the same direction as the wave propagation. Now we will study how to determine the amount of power that is flowing.

The average Poynting vector for a time-harmonic plane wave can be calculated from Equation 18.28, which we first saw last chapter:

$$\mathbf{S_{av}}(\mathbf{r}) = \frac{1}{2}Re[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^{\star}(\mathbf{r})] \quad \text{(Copy of Equation 18.28)}$$

Remember that the asterisk on **H(r)** means that you must take its complex conjugate before performing this calculation.

We can calculate the average power flow by taking the surface integral of this vector:

$$\boxed{P_{av} = \frac{1}{2}Re\left[\int_{\Delta s} \mathbf{E}(z,t) \times \mathbf{H}^{\star}(z,t) \bullet \mathbf{dS}\right]} \quad \text{(Equation 19.45)}$$

Since we now have the knowledge necessary to calculate **H**(z,t) from **E**(z,t), we can perform this calculation if we are given **E**(z,t) and the direction of propagation. As it turns out, we don't even need to know the values of the wave number, wavelength, frequency, or period—we only need to know the magnitude of **E**.

Example 19.4: If an electric field in a vacuum is known to obey the following equation, how much power would pass through a circular opening centered at the origin with z=0 and a radius of 2 meters?

$$\mathbf{E}(z,t) = 25cos(\omega t - kz)\mathbf{a_x} V/m$$

19.5 Summary

- Waves moving at a constant velocity to the left or right (or a combination of both) obey the wave equation:

$$E_y(z,t) = F(z-ct) + G(z+ct)$$

- Plane waves only depend on one spatial variable. This variable will be the direction of wave propagation, and it is different than the direction of wave polarization.

- Time-harmonic plane waves are a solution to the wave equation as long as the following criterion is met:

$$v_\phi = \frac{dz}{dt} = \pm c = \pm\frac{\omega}{k}$$

- The solution will have this form:

$$E_y(z,t) = ae^{j(\omega t - kz)} + be^{j(\omega t + kz)}$$

- If we know two of the values of v_ϕ, k, λ, T, f, and ω, we can find all the others.

- The direction of wave propagation is always the same as the direction of the Poynting vector, which is **E** x **H**.

- The ratio of the magnitudes of **E** and **H** can be found as follows:

$$Z_0 = \sqrt{\frac{\mu_o}{\epsilon_0}} = 377\Omega$$

- Given an expression to represent **E**, we can calculate **H** as follows:

$$H(z,t) = \frac{1}{Z_c}(\mathbf{a_k} \times \mathbf{E}(z,t))$$

- The average power flowing with a time-harmonic plane wave can be found as follows:

$$P_{av} = \frac{1}{2}Re\left[\int_{\Delta s} \mathbf{E}(z,t) \times \mathbf{H}^{\star}(z,t) \bullet \mathbf{dS}\right]$$