Uniform Plane Waves

1 The Helmholtz Wave Equation

Let's rewrite Maxwell's equations in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ exclusively. Let's assume the medium is lossless ($\sigma = 0$). Let's also assume we are operating in a source-free region, which means that there are no current sources ($\boldsymbol{J}_{src} = 0$) or free charge densities ($\rho_v = 0$). Maxwell's equations in phasor form then reduce to

$$\nabla \cdot \boldsymbol{E} = 0 \tag{1}$$
$$\nabla \cdot \boldsymbol{H} = 0 \tag{2}$$
$$\nabla \times \boldsymbol{E} = -j\omega\mu\boldsymbol{H} \tag{3}$$
$$\nabla \times \boldsymbol{H} = j\omega\varepsilon\boldsymbol{E}. \tag{4}$$

We note with the source-free assumption that the first two of Maxwell's equations are no longer independent relations, because they can be derived by taking the divergence of the last two equations. Therefore, we only really need the curl equations in this derivation.

Taking the curl of Equation (3) and substituting in Equation (4), we obtain

$$\nabla \times \nabla \times \boldsymbol{E} = -j\omega\mu\nabla \times \boldsymbol{H} = \omega^2\mu\varepsilon\boldsymbol{E}. \tag{5}$$

There is a vector identity that says that

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2\boldsymbol{A}, \tag{6}$$

i.e., the curl of a curl of a vector is equal to the gradient of the divergence of the vector minus the Laplacian of the vector. Using this identity, we obtain

$$\nabla^2\boldsymbol{E} + \omega^2\mu\varepsilon\boldsymbol{E} = 0, \tag{7}$$

an equation known as the vector Helmholtz equation for $\boldsymbol{E}$. Repeating for $\boldsymbol{H}$ yields

$$\nabla^2\boldsymbol{H} + \omega^2\mu\varepsilon\boldsymbol{H} = 0. \tag{8}$$

Let's focus our behaviour on the wave equation for $\boldsymbol{E}$ for now. Let's rewrite the equation as

$$\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = 0, \tag{9}$$

where $k = \sqrt{\omega^2\mu\varepsilon}$ is called the *wavenumber* (units of m^{-1}). The vector wave equation is an amazingly compact equation and is formidable when expanded: there are three Laplacians to expand in three dimensions (x, y, z in the case of a Cartesian coordinate system). We can expand the Laplacian operator to obtain

$$\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = \frac{\partial^2\boldsymbol{E}}{\partial x^2} + \frac{\partial^2\boldsymbol{E}}{\partial y^2} + \frac{\partial^2\boldsymbol{E}}{\partial z^2} + k^2\boldsymbol{E} = 0. \tag{10}$$

2 Case 1: Plane Waves in a Lossless Medium

We'll start off by assuming spatial variation in only one direction, say z. We will also assume that an electric field with only an x component. The resulting equation is

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0. \tag{11}$$

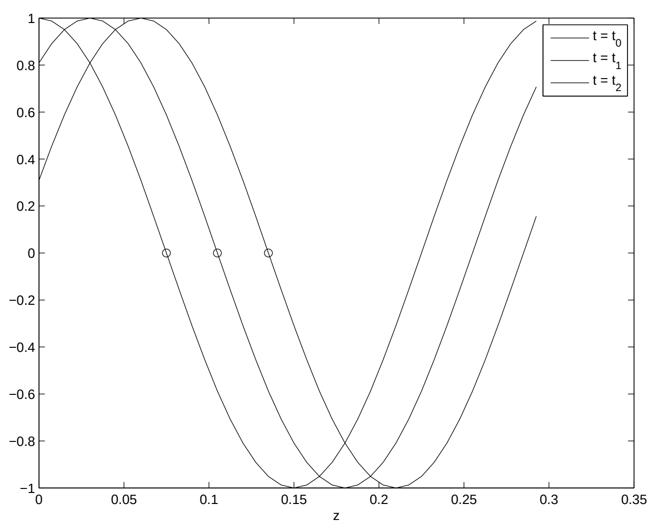
This is an ordinary differential equation with solutions

$$E_x(z) = E^+ e^{-jkz} + E^- e^{jkz}. \tag{12}$$

This is a phasor solution with a real-time representation of

$$\mathcal{E}(z,t) = E^+ \cos(\omega t - kz) + E^- \cos(\omega t + kz). \tag{13}$$

Each component of the wave solution is known as a *uniform plane wave* since the wave is uniform in the xy-plane, and the wave is infinite in extent. We see that the first term represents a wave travelling in the $+z$ direction. This is because if we consider a single point (a fixed phase point) on the cosine curve, and try to stay with that point (i.e., keep the argument of the cosine constant), as time increases, z must increase to maintain this condition. Hence, the wave travels in the $+z$ direction, while the second term refers to one travelling in the $-z$ direction.



The speed at which this phase point travels (the wave's *phase velocity*) is also determined using the constant phase condition:

$$\omega t - kz = K \Rightarrow z = \frac{\omega t - K}{k} \tag{14}$$

and

$$v_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}}. \tag{15}$$

If we use the permittivity and permeability of free space, we obtain the speed of light $c = 1/\sqrt{\varepsilon_0\mu_0} = 2.998 \times 10^8$ m/s.

The spatial period of the wave (the distance between two successive points of identical phase) is the wavelength of the wave. It can be measured at a fixed point in time. Since the argument of the cosine is periodic (i.e., it repeats every 2π),

$$[\omega t - kz] - [\omega t - k(z + \lambda)] = 2\pi, \tag{16}$$

and the wavelength λ is equal to

$$\lambda = \frac{2\pi}{k} = \frac{2\pi v_p}{\omega} = \frac{v_p}{f}. \tag{17}$$

The first equality is known as the *dispersion relation* since it relates the wavenumber to the wavelength. For the case considered here, there is a linear (proportional) relationship between the two.

Finally, let's find the relationship between the electric and magnetic fields for a plane wave. $\boldsymbol{H}$ can be found using the appropriate curl equation (3) from Maxwell's equations. The curl operator is quite straightforward since there is no variation in x and y:

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \left(\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}}\right)\hat{\boldsymbol{y}} + \left(\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}}\right)\hat{\boldsymbol{z}} & (18) \\ &= -\frac{\partial E_y}{\partial z}\hat{\boldsymbol{x}} + \frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}}, & (19) \end{aligned}$$

and since $E_y = 0$,

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \frac{\partial E_x}{\partial z}\hat{\boldsymbol{y}} = -j\omega\mu\boldsymbol{H} & (20) \\ H_y &= \frac{j}{\omega\mu}\frac{\partial E_x}{\partial z}. & (21) \end{aligned}$$

Evaluating the derivative,

$$\begin{aligned} H_y &= \frac{j}{\omega\mu}(-jk)\left(E^+e^{-jkz} - E^-e^{jkz}\right) & (22) \\ &= \frac{k}{\omega\mu}\left(E^+e^{-jkz} - E^-e^{jkz}\right). & (23) \end{aligned}$$

Since $\frac{k}{\omega\mu} = \frac{\omega\sqrt{\mu\varepsilon}}{\omega\mu} = \sqrt{\frac{\varepsilon}{\mu}}$, we can write the equation as

$$H_y = \frac{1}{\eta}\left[E^+e^{-jkz} - E^-e^{jkz}\right], \tag{24}$$

where

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{25}$$

is defined as the *wave impedance* of the plane wave in ohms. Hence, $\boldsymbol{E}$ and $\boldsymbol{H}$ for each travelling wave are related through a simple multiplication operation: the magnetic field is equal to the electric field divided by the wave impedance. The sign the product depends on the direction of power flow according to the sign of the Poynting vector. This is analogous to the relationship between voltage and current in guided-wave (transmission line) scenarios whereby we use the *characteristic impedance* to relate the two quantities. Note that the wave impedance of free space is $\sqrt{\frac{\mu_o}{\varepsilon_0}} = 377\ \Omega$.

3 Case 2: Plane Waves in a Lossy Medium

In a lossless medium, $\sigma = 0$. However, a nonzero conductivity changes the Helmholtz equation slightly and hence the plane wave solution. Maxwell's curl equation for $\boldsymbol{H}$ becomes

$$\begin{aligned} \nabla \times \boldsymbol{H} &= j\omega \boldsymbol{D} + \boldsymbol{J} && (26) \\ &= j\omega\varepsilon \boldsymbol{E} + \sigma \boldsymbol{E} && (27) \\ &= jw\varepsilon \boldsymbol{E}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right). && (28) \end{aligned}$$

Note the quantity $\sigma/j\omega\varepsilon$ is the ratio of the conduction current to the displacement current. The quantity $\tan\delta = \frac{\sigma}{\omega\varepsilon}$ is often called the *loss tangent* of the medium. The angle δ is the angle by which the displacement current leads the total current density.

The resulting wave equation for $\boldsymbol{E}$ becomes

$$\nabla^2 \boldsymbol{E} + \omega^2\mu\varepsilon \boldsymbol{E}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) = 0. \tag{29}$$

Then effectively, the expression $k^2 = \omega^2\mu\varepsilon$ is been replaced with $\sqrt{\omega^2\mu\varepsilon[1 - j(\sigma/\omega\varepsilon)]}$. Let us define a complex propagation constant of the medium as

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}. \tag{30}$$

The wave equation becomes

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0, \tag{31}$$

which has solutions

$$E_x(z) = E^+ e^{-\gamma z} + E^- e^{\gamma z}. \tag{32}$$

Examining the positive travelling wave component, we see it has the form

$$E^+ e^{-\gamma z} = E^+ e^{-\alpha z} e^{-j\beta z}, \tag{33}$$

which in the time domain is represented as

$$E^+ e^{-\alpha z}\cos(\omega t - \beta z). \tag{34}$$

We see that it is similar to the positive travelling wave in a lossless medium, with the exception that there is an exponential damping factor in front of the harmonic term, indicating that the wave is attenuated as z increases. A similar damping exists for the negative travelling wave. Hence, in a lossy medium, the wave is dissipated in the medium and is attenuated with distance.

If the loss is removed ($\sigma = 0$), $\alpha = 0$ and the solution is the same as the lossless case with $\gamma = jk = j\omega\sqrt{\mu\varepsilon}$. Therefore, $k = \beta$ for lossless media and the two terms are often used interchangeably.

The wave impedance for the lossy case is evaluated from Equation 21 and is found as follows:

$$\begin{aligned} H_y &= \frac{j}{\omega\mu}(-\gamma)\left(E^+ e^{-\gamma z} - E^- e^{\gamma z}\right) & (35) \\ \eta &= \frac{j\omega\mu}{\gamma}, & (36) \end{aligned}$$

which we note is a complex quantity since γ is complex.

4 Case 3: Plane Waves in a Good Conductor

In the case of a medium with high conductivity such as a metal, $\frac{\sigma}{\omega\varepsilon} \gg 1$ and

$$\begin{aligned} \gamma &= j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}} \approx j\omega\sqrt{\mu\varepsilon}\sqrt{-j\frac{\sigma}{\omega\varepsilon}} & (37) \\ &= \sqrt{j}\sqrt{\omega\sigma\mu} & (38) \\ &= (1+j)\sqrt{\frac{\omega\sigma\mu}{2}}, & (39) \end{aligned}$$

since $\sqrt{j} = 1\angle 45^\circ = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$. The damping introduced by the $e^{-\alpha z}$ clearly dominates now due to the magnitude of α (the real part of γ). In fact, the plane wave will only penetrate the medium to a depth

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\sigma\mu}}, \tag{40}$$

which is defined as the *skin depth* of the conductor. At this depth, the amplitude of the plane wave has decayed by an amount $1/e$ of 36.8% of its original value. At high frequencies this depth is vary small and hence waves propagate along the "skin" of the conductor. This has practical importance since the ohmic loss introduced by a metal can be exaggerated by the fact that microwave signals only propagate in a portion of the conductor, versus the entire cross section.

5 General Plane Wave Solutions

We now re-consider the lossless case (Case 1), which developed a plane wave that was 1) polarized in the $\hat{\boldsymbol{x}}$ direction, and 2) propagated along the z-axis. We now want to consider a general plane wave that has arbitrary linear polarization, and travels in any direction. To solve the Helmholtz equation

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + k^2 \boldsymbol{E} = 0, \tag{41}$$

we first notice that it reduces to three scalar equations,

$$\nabla^2 E_x + k^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \tag{42a}$$

$$\nabla^2 E_x + k^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} + k^2 E_y = 0 \tag{42b}$$

$$\nabla^2 E_x + k^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0. \tag{42c}$$

The solution to the first equation, for example, is assumed to be written in the form

$$E_x(x, y, z) = f(x)g(y)h(z), \tag{43}$$

that is, the wave solution can be expressed as a product of three independent functions, each acting on a separate spatial variable. We are free to make this assumption and if any inconsistencies arise in assuming such a form, we could attempt another form for the solution. Using the assumed form, the wave equation for E_x becomes

$$gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} + k^2 fgh = 0. \tag{44}$$

Since f, g, and h are only functions of one spatial variable each, we can replace the partial derivatives with ordinary derivatives. Doing this and dividing by fgh,

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} + k^2 = 0 \tag{45}$$

or

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -k^2. \tag{46}$$

Since each term on the left hand side is dependent on only one spatial variable, the sum of these terms can only equal $-k^2$ if each term is a constant. Hence, we write

$$\frac{d^2 f}{dx^2} = -k_x^2 f \tag{47a}$$

$$\frac{d^2 g}{dy^2} = -k_y^2 f \tag{47b}$$

$$\frac{d^2 h}{dz^2} = -k_z^2 f \tag{47c}$$

We have already solve versions of these equations in our 1D example earlier. The only additional constraint then, is that in addition our solution satisfying all three equations simultaneously, the wavenumber components must also satisfy

$$k_x^2 + k_y^2 + k_z^2 = k^2, \tag{48}$$

so that (46) is satisfied. This relation is called the *consistentency relation* or *dispersion relation.* Most generally, we can write a solution satisfying both (46) and the dispersion relation as

$$E_x(x,y,z) = Ae^{-j(k_x x + k_y y + k_z z)}. \tag{49}$$

Let's simplify this expression by defining a vector wavenumber,

$$\boldsymbol{k} = k_x \hat{\boldsymbol{x}} + k_y \hat{\boldsymbol{y}} + k_z \hat{\boldsymbol{z}} = k\,\hat{\boldsymbol{n}}. \tag{50}$$

If we define a position vector

$$\boldsymbol{r} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}, \tag{51}$$

then we can express E_x as

$$E_x(x,y,z) = Ae^{-j\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{52}$$

Similarly,

$$E_y(x,y,z) = Be^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \tag{53}$$
$$E_z(x,y,z) = Ce^{-j\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{54}$$

Finally, let's combine all the field components into a vector expression for the electric field,

$$\boldsymbol{E} = (A\hat{\boldsymbol{x}} + B\hat{\boldsymbol{y}} + C\hat{\boldsymbol{z}})e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{55}$$

With a potential solution for the Helmholtz equation, let us check it satisfies the divergence relation in Maxwell's equations by evaluating $\boldsymbol{\nabla}\cdot\boldsymbol{E}$. To do this, we will make the use of the following vector identity:

$$\nabla\cdot f\boldsymbol{A} = \boldsymbol{A}\cdot\nabla f + f\nabla\cdot\boldsymbol{A} \tag{56}$$

$$\nabla\cdot\boldsymbol{E} = \boldsymbol{E}_0\cdot\nabla\left(e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\right) + e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\left(\nabla\cdot\boldsymbol{E}_0\right) \tag{57}$$
$$= \boldsymbol{E_0}\cdot(-j\boldsymbol{k})\,e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} = 0. \tag{58}$$

From this equation, we can see that that $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$ – that is, the wavenumber $\boldsymbol{k}$ and polarization of the electric field are *orthogonal*. Since $\boldsymbol{k}$ indicates the direction the plane wave is travelling, an important conclusion is that the electric field is orthogonal to the direction of propagation.

What about the magnetic field? Knowing $\boldsymbol{E} = \boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}$, we can find the magnetic field by evaluating

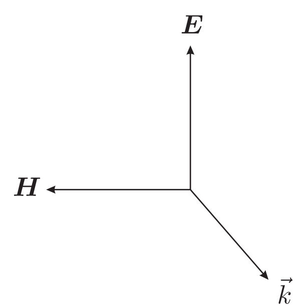
$$\boldsymbol{H} = \frac{j}{\omega\mu}\nabla\times\boldsymbol{E}, \tag{59}$$

with the assistance of the vector identity

$$\nabla\times f\boldsymbol{A} = (\nabla f)\times\boldsymbol{A} + f\nabla\times\boldsymbol{A}. \tag{60}$$

$$\begin{aligned}
\boldsymbol{H} &= \frac{j}{\omega\mu}\left[\left(\nabla e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\right)\times\boldsymbol{E}_0 + e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\left(\nabla\times\boldsymbol{E}_0\right)\right] && (61)\\
&= \frac{j}{\omega\mu}(-j\boldsymbol{k})e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\times\boldsymbol{E}_0 && (62)\\
&= \frac{k\,\hat{\boldsymbol{n}}}{\omega\mu_0}e^{-j\boldsymbol{k}\cdot\boldsymbol{r}}\times\boldsymbol{E}_0 && (63)\\
&= \frac{1}{\eta}\hat{\boldsymbol{n}}\times\boldsymbol{E}_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} && (64)\\
&= \frac{1}{\eta}\hat{\boldsymbol{n}}\times\boldsymbol{E}. && (65)
\end{aligned}$$

This is not a surprising result: $\boldsymbol{E}$ and $\boldsymbol{H}$ are related through the wave impedance, and are perpendicular to each other. Furthermore, $\boldsymbol{H}$ is also perpendicular to the direction of propagation. Hence, $\boldsymbol{E}$, $\boldsymbol{H}$, and $\boldsymbol{k}$ all form a triplet related through the right-hand rule: that the direction of $\boldsymbol{k}$ is equal to the direction of $\boldsymbol{E}\times\boldsymbol{H}$. The right-handed triplet is illustrated below.



Since the electric and magnetic fields are orthogonal to the direction of propagation, we call such a wave a *transverse electromagnetic wave*, which you may have heard about in an earlier field and waves course. This contrasts to transverse electric (TE) and transverse magnetic (TM) wave solutions that also exist in wave problems.