4. Two Dimensional Geometric Transformations

Contents

Introduction

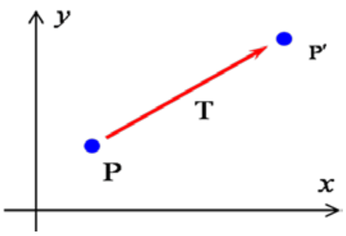
- Transformations are a fundamental part of computer graphics.
- Transformations are used to position objects, to shape objects, to change viewing positions, and even to change how something is viewed. There are 5 main types of transformations that can be performed in 2 dimensions:
- translations
- scaling
- rotation
- reflection
- shearing

4.1 Translation

A translation moves all points in an object along the same straight-line path to new positions.

The path is represented by a vector, called the translation or shift vector.

We can write the components:



$$Px' = Px + Tx$$

$$Py' = Py + Ty$$

or in matrix form: P' = P + T

$$x' = x + t_x, \quad y' = y + t_y$$

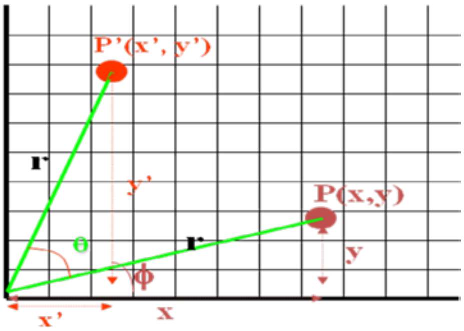
$$\mathbf{P} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{P}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{T}$$

4.2 Rotation

point p(X,Y) is to be rotated about the origin by angle theta to location p'(X',Y')

Let us consider the diagram



- We use some geometric concept here

$$\Rightarrow cos\emptyset = {}^{x}/_{r} \quad , sin\,\emptyset = {}^{y}/_{r}$$

$$x = rcos\emptyset \; , y = rsin\emptyset$$

$$\Rightarrow \cos(\emptyset + \theta) = {}^{x'}/_{r}$$

$$x' = rcos(\emptyset + \theta)$$

$$x' = rcos\emptyset cos\theta - rsin\emptyset sin\theta$$

$$x' = xcos\theta - ysin\theta$$

$$\Rightarrow \sin(\emptyset + \theta) = {}^{y'}/_{r}$$

$$y' = rsin(\emptyset + \theta)$$

$$y' = rcos\emptyset sin\theta - rsin\emptyset cos\theta$$

$$y' = xsin\theta + ycos\theta$$

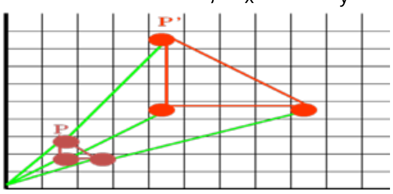
in matrix form: P' = R X P

$$Where\ R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

4.3 Scaling

- Scaling changes the size of an object and involves two scale factors, S_x and S_y for the x- and y- coordinates respectively.
- Scales are about the origin.
- Consider the diagram for generating equation
- We can write the components as:



$X' = Sx.X$

$Y' = Sy.Y$

P' = S • P ,The matrix represented form is

$$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

The matrix form is

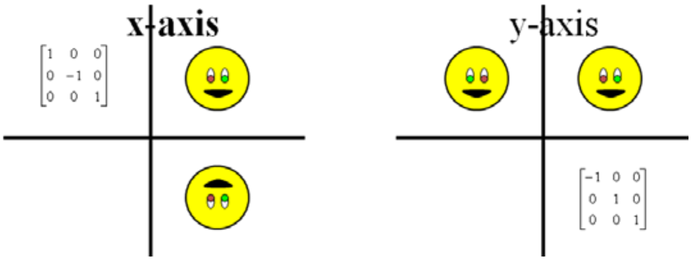
$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

- If the scale factors are in between 0 and 1 ➔ the points will be moved closer to the origin ➔ the object will be smaller.
- If the scale factors are larger than 1 ➔ the points will be moved away from the origin ➔ the object will be larger.

4.4 Reflection

- A reflection is a transformation that produces the mirror image of an object.
- The mirror image for 2D reflection is generated relative to an axis of reflection by rotating 180^0.
- We can choose an axis of reflection in a xy-plane and perpendicular to the xy-plane.
- The matrix representation with homogeneous coordinate form along x-axis is

Reflection along x-axis and y-axis

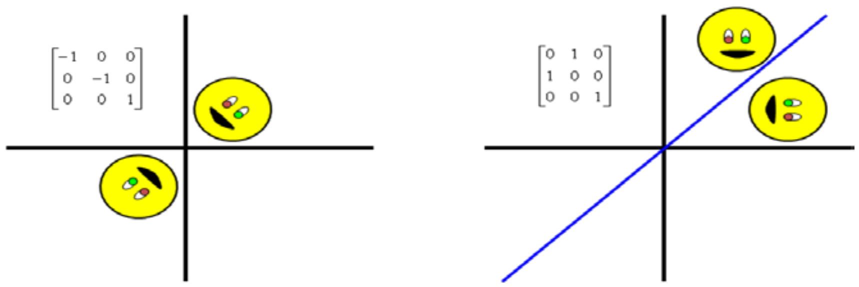


Reflection along x-axis is $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

Reflection along y-axis is representation in matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection:origin and line x=y



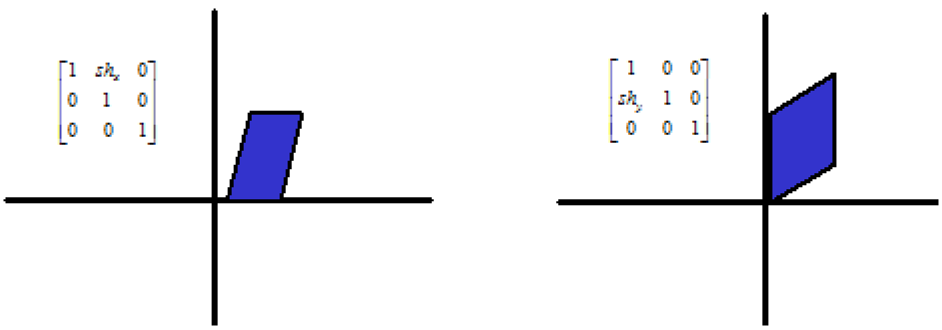
4.5 Shear

- The matrix expression for the shearing transformation of a position P = (*x*, *y*,) to produce x-axis shear and y-axis shear.
- Share along x-axis is represented in matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Share along y-axis is represented in matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



4.6 Homogenous Coordinates

- The fact that all the points along each line can be mapped back to the same point in 2D gives this coordinate system its name – homogeneous coordinates**.**
  Make displacement linear with homogeneous coordinates$(x, y) \Rightarrow (x, y, 1)$

- Transformation turns into 3X3 matrix. Very big advantage –All transformations are concatenated by matrix multiplication Using homogenous coordinates we will do all transformations

2D Translation
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{T}(t_x, t_y) \cdot \mathbf{P}$$

2D Rotation
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{R}(\theta) \cdot \mathbf{P}$$

2D Scaling
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \quad \mathbf{P}' = \mathbf{S}(S_x, S_y) \cdot \mathbf{P}$$

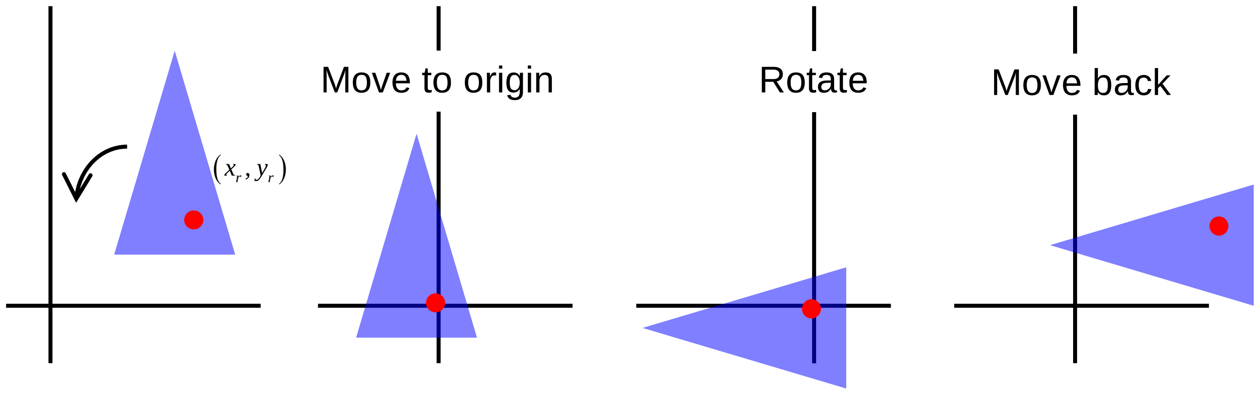
4.7 Composite Transformation Matrix

Forming products of transformation matrix is referred to as concatenation or composition. We from composite transformation by multiplying matrices in order to form left to right.

General Fixed-Point Rotation:-

- Arrange the transformation matrices in order from right to left.
- General Pivot- Point Rotation
  - Operation :-
    1. Translate (pivot point is moved to origin)
    2. Rotate about origin
    3. Translate (pivot point is returned to original position)

4. Consider the following diagram



Original position of the object & pivot point — Translate the object so that pivot point — Rotate the object about origin — Translate the object so that the point at the origin

T(pivot) • R(θ) • T(–pivot)

The transformation matrix is

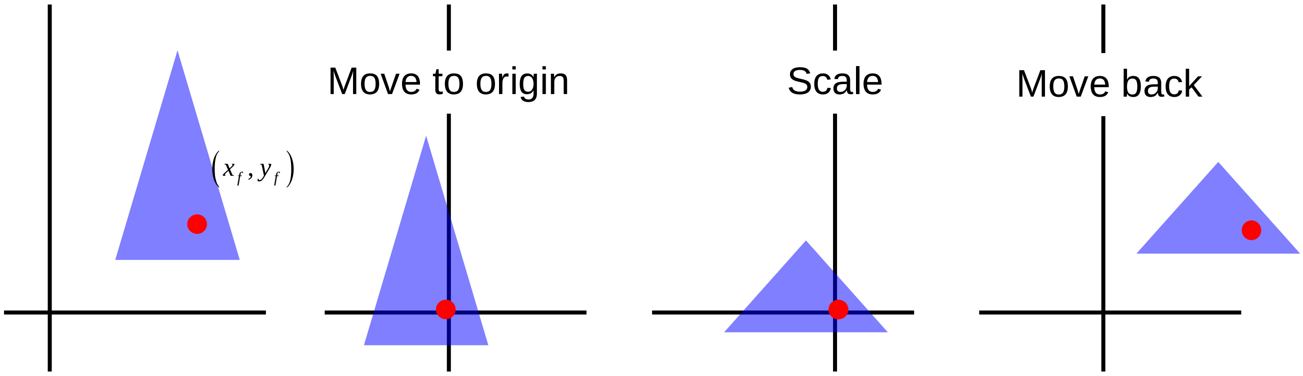
$$\begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta)+y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta)-x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

General Fixed-Point Scaling

Operations:

1. Translate (fixed point is moved to origin)
2. Scale with respect to origin
3. Translate (fixed point is returned to original position)



Original position of the object & pivot point

Translate the object so that pivot point about origin

Scale the object

Translate the object so that the point at the origin

T(fixed) • S(scale) • T(–fixed)

The transformation matrix is

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & x_f\left(1-S_x\right) \\ 0 & S_y & y_f\left(1-S_y\right) \\ 0 & 0 & 1 \end{bmatrix}$$