Mobius Transformation
(or)
Bilinear transformation

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ where a, b, c, d are complex numbers, is called a bilinear transformation.

Formula:

Bilinear transformation of z_1, z_2, z_3 into w_1, w_2, w_3 is gvn. by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

1. Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Soln.:

Given $z_1 = 0,\ z_2 = -i,\ z_3 = -1$

$w_1 = i,\ w_2 = 1,\ w_3 = 0$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(1-0)}{(w-0)(1-i)} = \frac{(z-0)(-i+1)}{(z+1)(-i-0)}$$

$$\frac{w-i}{w-wi} = \frac{z(1-i)}{(-i)(z+1)}$$

$$\frac{w-i}{w-wi} = \frac{z-zi}{-iz-i}$$

$$(w-i)(-iz-i) = (z-zi)(w-wi)$$

$$-wzi - wi - z - 1 = wz - wzi - wzi - wz$$

$$-wzi - wi + wzi + wzi = z+1$$

$$wzi - wi = z+1$$

$$wi(z-1) = z+1$$

$$\omega = \frac{1}{i}\,\frac{z+1}{z-1}$$

$$= \frac{i}{i^2}\,\frac{z+1}{z-1}$$

$$\omega = -i\left(\frac{z+1}{z-1}\right)$$

2]. Find the bilinear transformation which maps $z = 1, i, -1$ to $\omega = i, 0, -i$.

Soln.

Given

$z_1 = 1,\quad z_2 = i,\quad z_3 = -1$

$\omega_1 = i,\quad \omega_2 = 0,\quad \omega_3 = -i$

The bilinear transformation is,

$$\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(\omega-i)(0+i)}{(\omega+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{i(\omega-i)+(-i)(\omega+i)}{i(\omega-i)-(-i)(\omega+i)}$$

$$\frac{(\omega-i)}{-(\omega+i)} = \frac{z-1}{z+1}\left(\frac{1+i}{-1+i}\right)$$

$$= \left(\frac{z-1}{z+1}\right)\frac{(1+i)(-1-i)}{(-1+i)(-1-i)}$$

$$\omega = \frac{-iz-1}{iz-1}$$

$$= \left(\frac{z-1}{z+1}\right)\frac{-1-i-i+1}{(-1)^2-i^2}$$

$$= \frac{z-1}{z+1}\left(\frac{-2i}{2}\right)$$

$$-\left[\frac{\omega-i}{\omega+i}\right] = -\left[\frac{z-1}{z+1}\right] i$$

$$\frac{\omega-i}{\omega+i} = \frac{iz-i}{z+1}$$

By componendo & dividendo,

$$\frac{(\omega-i)+(\omega+i)}{(\omega-i)-(\omega+i)} = \frac{(iz-i)+(z+1)}{(iz-i)-(z+1)}$$

$$\Rightarrow \frac{2w}{-2i} = \frac{z(i+1)+(1-i)}{z(i-1)-(1+i)}$$

$$w = -i\,\frac{z(1+i)+(1-i)}{z(i-1)-(1+i)}$$

3] Find the bilinear transformation which maps $-1, -i, 1$ in z-plane to $\infty, i, 0$ in w-plane respectively.

Soln.

Given $z_1 = -1,\ z_2 = -i,\ z_3 = 1$

$w_1 = \infty,\ w_2 = i,\ w_3 = 0$

The Bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w_1\left(\frac{w}{w_1}-1\right)(w_2-w_3)}{(w-w_3)\,w_1\left(\frac{w_2}{w_1}-1\right)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(0-1)(i-0)}{w(0-1)} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$\frac{-i}{-w} = \frac{(z+1)(-i-1)}{(z-1)(-i+1)}$$

$$w = i\,\frac{(z-1)(-i+1)}{(z+1)(-i-1)}$$

$$= i\,\frac{[-iz+z+i-1]}{(-iz-z-i-1)}$$

$$= \frac{z+iz-1-i}{-iz-z-i-1}$$

$$= \frac{(z-1)+i(z-1)}{(-z-1)+i(-z-1)}$$

$$= \frac{(z-1)(1+i)}{(-z-1)(1+i)}$$

$$w = \frac{1-z}{1+z}$$

$$w = \frac{1-z}{z+1}$$

4]. Find the bilinear transformation which maps $\infty, i, 0$ to $0, i, \infty$.

Soln.

Given $z_1 = \infty,\ z_2 = i,\ z_3 = 0$

$w_1 = 0,\ w_2 = i,\ w_3 = \infty$

The bilinear transformation is,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)\, w_3\left(\frac{w_2}{w_3}-1\right)}{w_3\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\, z_1\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{\left(\frac{w}{w_3}-1\right)(w_2-w_1)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z-z_3)\left(\frac{z_2}{z_1}-1\right)}$$

$$\frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$w = \frac{1}{z}$$

Soln.

Given $z_1 = 0,\ z_2 = 1,\ z_3 = \infty$

$w_1 = -5,\ w_2 = -1,\ w_3 = 3$

The bilinear transformation is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)\, z_3\left(\frac{z_2}{z_3}-1\right)}{z_3\left(\frac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+5)(-1-3)}{(w-3)(-1+5)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$\frac{(w+5)(-4)}{(w-3)(4)} = \frac{-z}{-1}$$

$$\frac{-(w+5)}{w-3} = \frac{z}{1}$$

$$-(w+5) = z(w-3)$$

$$-w-5 = wz - 3z$$

$$wz + w = 3z - 5$$

$$w(z+1) = 3z - 5$$

$$w = \frac{3z-5}{}$$