Necessary condition for f(z) to be analytic:

If $w = f(z) = u + iv$ is an analytic function, then Cauchy_Riemann eqns are satisfied.

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\Rightarrow u_x = v_y$ and $v_x = -u_y$

Sufficient condition for Analytic function:

If the partial derivatives u_x, u_y, v_x and v_y are all continuous and $u_x = u_y$ and $u_y = -v_x$, then the function is analytic.

1. Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln.

Given $f(z) = \bar{z} = x - iy$

$u + iv = x - iy$

$\Rightarrow u = x$ and $v = -y$

$u_x = 1$ $\quad v_x = 0$

$u_y = 0$ $\quad v_y = -1$

Here $u_x \neq v_y$ and $u_y = -v_x$

Hence C_R eqns are not satisfied.

$\Rightarrow f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

2. Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Soln.

Let $f(z) = 2xy + i(x^2 - y^2)$

$u + iv = 2xy + i(x^2 - y^2)$

$\Rightarrow u = 2xy$ and $v = x^2 - y^2$

$u_x = 2y$ $\quad$ $v_x = 2x$

$u_y = 2x$ $\quad$ $v_y = -2y$

$\Rightarrow u_x \neq v_y$ and $u_y \neq -v_x$

C.R eqns. are not satisfied.

Hence $f(z)$ is not an analytic function.

3. Let $f(z) = z^3$ be analytic. Justify.

Soln.

Let $f(z) = z^3$

$$u + iv = (x + iy)^3$$

$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3$$

$$u+iv = [x^3 - 3xy^2] + i[3x^2y - y^3]$$

$\Rightarrow u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

$u_x = 3x^2 - 3y^2$ $v_x = 6xy$

$u_y = -6xy$ $v_y = -3y^2 + 3x^2$

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

CR eqns are satisfied.

Hence $f(z)$ is analytic.

4. Find the constants a, b, c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Soln.

Let $f(z) = x + ay + i(bx + cy)$

$u + iv = x + ay + i(bx + cy)$

Here $u = x + ay$ and $v = bx + cy$

$u_x = 1$ $v_x = b$

$u_y = a$ $v_y = c$

Since $f(z)$ is analytic.

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ $a = -b$

$\therefore a = -b$ and $c = 1$.

5. Check whether the function $w = \sin z$ is analytic (or) not.

$\cos iy = \cosh y$
$\sin iy = i \sinh y$

Soln.

Let $w = f(z) = \sin z$

$u + iv = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sinh y$

Here $u = \sin x \cosh y$ and $v = \cos x \sinh y$

$u_x = \cos x \cosh y$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $v_y = \cos x \cosh y$

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

CR eqns are satisfied.
Also the 4 partial derivatives are continuous.
Hence the function is analytic.

Properties of Analytic function:

i) $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is known as the laplace eqn in two dimension.

Property 1:
The real and imaginary part of an analytic fn. $w = u + iv$ satisfies Laplace eqn.

Proof:
Let $w = f(z) = u + iv$ be analytic.
To prove u and v satisfies laplace eqn.
i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Since $f(z)$ is analytic.
$\Rightarrow u_x = v_y$ and $u_y = -v_x \rightarrow (1)$
Differentiate (1) partially w.r. to x and y,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \rightarrow (2)$$

and $$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \rightarrow (3)$$

$$(2)+(3) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$\Rightarrow u$ satisfies Laplace eqn.

Differentiate (1) partially w.r. to y and x

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \Big| \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \to (4) \qquad \Big| \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \to (5)$$

$$(4) + (5) \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}$$

$$= 0$$

$\Rightarrow v$ satisfies Laplace eqn.

Hence u & v satisfies Laplace eqn.

Property 2:

Show that an analytic function with constant real part is constant.

Proof:

Let $f(z)$ be analytic function.

$\Rightarrow u_x = v_y$ and $u_y = -v_x \longrightarrow (1)$

Given $u = \text{constant} = c$ (say)

Differentiate partially w.r. to x & y,

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

i.e., $u_x = 0$ $\qquad$ $u_y = 0$

From (1) $v_y = 0$ $\qquad$ $-v_x = 0 \Rightarrow v_x = 0$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 0 + i(0)$$

$$f'(z) = 0$$

$$\Rightarrow f(z) = c$$

$\therefore f(z)$ is constant.

Property 3:

An analytic function with constant modulus is analytic.