Fourier Transform Pair:

The fourier transform of $f(x)$ is gvn. by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \rightarrow (1)$$

Then the function $f(x)$ is the Inverse fourier transform of $F(s)$ is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \rightarrow (2)$$

The above eqns. (1) and (2) are jointly called Fourier transform Pair.

Self Reciprocal function:

If the fourier transform of $f(x)$ is equal to $F(s)$, then $f(x)$ is said to be reciprocal function under fourier transform.

i.e., $F[f(x)] = F(s)$

Eg: $F[e^{-x^2/2}] = e^{-s^2/2}$

Parseval's Identity or Rayleigh's Theorem:

If $F(s)$ is the fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

Results:

i). $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ ii). $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

Problems :

1]. Find the FT of $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases}$

Soln. :

$$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2 \int_{0}^{a} \cos sx\, dx + i\, 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin sa}{s}$$

2]. Find the FT of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

Soln. :

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x \cos sx\, dx + i \int_{-a}^{a} x \sin sx\, dx \right]$$

$x \cos sx \to$ odd

$x \sin sx \to$ even

$$= \frac{1}{\sqrt{2\pi}}\left[0 + 2i\int_0^a x \sin sx \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{2i\left[x\left(\frac{-\cos sx}{s}\right) - 1\left(\frac{-\sin sx}{s^2}\right) + 0\right]_0^a\right\}$$

$$= \frac{1}{\sqrt{2\pi}}2i\left[\left(\frac{-a\cos sa}{s} + \frac{\sin sa}{s^2}\right) - 0\right]$$

$$= \sqrt{\frac{2}{\pi}}\,i\left[\frac{\sin sa - sa\cos sa}{s^2}\right]$$

3]. Show that fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$

and hence find that $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right]$. Hence deduce that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \pi/4$. Using P.I. Show that

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$

Soln. :

$$f(x) = \begin{cases} a^2 - x^2, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} (a^2 - x^2)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} \underbrace{(a^2 - x^2)}_{\text{even}} \underbrace{\cos sx}_{\text{even}} \, dx + i\int_{-a}^{a} \underbrace{(a^2 - x^2)}_{\text{even}} \underbrace{\sin sx}_{\text{odd}} \, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[2\int_0^a (a^2-x^2)\cos sx\,dx + i(0)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) - 0\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2x\frac{\cos sx}{s^2}\Big]_0^a + (a^2-x^2)\frac{\sin sx}{s} + 2\frac{\sin sx}{s^3}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2} + \frac{2\sin sa}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - sa\cos sa}{s^3}\right]$$

Put $a=1$, $F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

i). Using Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\left[\cos sx - i\sin sx\right]ds$$

$$= \frac{2}{\pi}\,2\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds$$

Put $x=0$, $f(0) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds$

$$\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)ds = \frac{\pi}{4}f(0) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \pi/4$$

ii). Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left\{ 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)\right\}^2 ds$$

$$a^2 - 2ab + b^2$$

$$2\int_{0}^{1} (1 + x^4 - 2x^2)\, dx = \frac{8\times 2}{\pi} \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi}\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - 2\frac{x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15+3-10}{15}\right]$$

$$= \frac{16}{15}$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right)$$

$$\int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

Q. Find the FT of $f(x) = \begin{cases} a - |x|, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ and deduce that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt$

Soln. :

$$f(x) = \begin{cases} a-|x|, & -a < x < a \\ 0, & -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a-x) \cos sx \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[(a-x) \frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos sx}{s^2} \right]_0^a$$

$$= -\sqrt{\frac{2}{\pi}} \left[\frac{\cos sa}{s^2} - \frac{1}{s^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos sa}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2 \sin^2 \frac{sa}{2}}{s^2} \qquad \cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos\theta = 1 - 2\sin^2 \theta/2$$

i). using IFT

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{s^2} \sqrt{\frac{2}{\pi}} \sin^2 \frac{sa}{2} \, e^{-isx} ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{s^2} \sin^2 \frac{sa}{2} [\cos sx - i \sin sx] ds$$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{1}{s^2} \sin^2 \frac{sa}{2} \cos sx \, ds$$

Put $x=0,\ a=2$

$$f(0) = \frac{4}{\pi}\int_0^\infty \frac{\sin^2 s}{s^2}\, ds$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(2-|0|)$$

$$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2} \qquad \text{'s' is dummy variable.}$$

ii). Parseval's Identity:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-2}^{2} [2-|x|]^2\, dx = \int_{-\infty}^{\infty} 4\,\frac{2}{\pi}\,\frac{\sin^4 \frac{sa}{2}}{s^4}\, ds$$

$$2\int_0^2 (2-x)^2\, dx = \frac{8}{\pi}\, 2\int_0^\infty \left[\frac{\sin^4 \frac{sa}{2}}{s}\right]^4 ds$$

$$2\left[\frac{(2-x)^3}{-3}\right]_0^2 = \frac{16}{\pi}\int_0^\infty \left[\frac{\sin \frac{sa}{2}}{s}\right]^4 ds$$

$$-\frac{2}{3}[0-8] = \frac{16}{\pi}\int_0^\infty \left[\frac{\sin s}{s}\right]^4 ds \qquad \because a=2$$

$$\frac{16}{3}\,\frac{\pi}{16} = \int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt \qquad (\because s \text{ is dummy var})$$

$$\int_0^\infty \left[\frac{\sin t}{t}\right]^4 dt = \frac{\pi}{3}$$

3]. Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$

and deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$

Soln. :

$$f(x) = \begin{cases} 1-|x|, & -1<x<1 \\ 0, & -\infty<x<-1 \text{ and } 1<x<\infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)\,(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} \underbrace{(1-|x|)}_{\text{even}} \underbrace{\cos sx}_{\text{even}}\, dx + i \int_{-1}^{1} \underbrace{(1-|x|)}_{\text{even}} \underbrace{\sin sx}_{\text{odd}}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^1 (1-x) \cos sx\, dx + 0 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\frac{\sin sx}{s} - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} + \frac{1}{s^2} \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

IFT $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right] [\cos sx - i \sin sx]\, ds$$

$$\frac{1-\cos(-s)}{(-s)^2} = \frac{1-\cos s}{}$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty}\left[\frac{1-\cos s}{s^2}\right]\cos sx\, ds - i\int_{-\infty}^{\infty}\left[\frac{1-\cos s}{s^2}\right]\sin sx\, ds\right]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\, ds - i(0)\right]$$

$\left(\frac{1-\cos s}{s^2}\right)\cos sx \to$ Even

$\left(\frac{1-\cos s}{s^2}\right)\sin sx \to$ odd

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{2\sin^2 s/2}{s^2}\right)\cos sx\, ds$$

Put $x=0$,

$$f(0) = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s/2}{s}\right)^2 ds$$

$$\int_{0}^{\infty}\left(\frac{\sin s/2}{s/2}\right)^2 ds = \pi f(0) = \pi/4 \qquad \because f(0) = 1-0 = 1$$

Put $t = s/2$ | when $s=0 \Rightarrow t=0$, $s=\infty \Rightarrow t=\infty$

$dt = \frac{ds}{2} \Rightarrow ds = 2\,dt$

$$\int_{0}^{\infty}\left(\frac{\sin t}{2t}\right)^2 2\,dt = \pi/4$$

$$\Rightarrow \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Using Parseval's Identity

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-|x|)^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2 ds$$

$$2\int_{0}^{1} (1-x)^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin^2 s/2}{s^2}\right)^2 ds$$

$$= \frac{8}{\pi}\int_{-\infty}^{\infty} \frac{\sin^4 s/2}{s^4} ds$$

$$2\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{16}{\pi}\int_{0}^{\infty} \frac{\sin^4 s/2}{s^4} ds$$

$$-\frac{2}{3}[0-1] = \frac{1}{\pi}\int_{0}^{\infty} \frac{\sin^4 s/2}{(s/2)^4} ds$$

$$\frac{2}{3} = \frac{1}{\pi}\int_{0}^{\infty} \left(\frac{\sin s/2}{s/2}\right)^4 ds$$

Take $t = s/2$

$dt = ds/2 \Rightarrow ds = 2dt$

when

$s=0 \Rightarrow t=0$

$s=\infty \Rightarrow t=\infty$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 2\,dt = \frac{2\pi}{3}$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{2\pi}{6}$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^t dt = \frac{\pi}{3}$$

Problems on FT:

4]. Find the Fourier transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$

Deduce that i). $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$

ii). $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

Soln. :

we already find $F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)$

IFT

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left(\frac{\sin as}{s}\right) (\cos sx - i \sin sx)\, ds$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right) \sin sx\, ds\right]$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds - i(0)$$

$$\int_{0}^{\infty} \left(\frac{\sin as}{s}\right) \cos sx\, ds = \frac{\pi}{2} f(x)$$

Put $x = 0,\ a = 1$

$$\int_{0}^{\infty} \left(\frac{\sin as}{s}\right) ds = \frac{\pi}{2} f(0) \qquad f(0) = 1$$

$$= \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty} \left(\frac{\sin t}{t}\right) dt = \pi/2 \qquad (\because s \text{ is dummy variable})$$

ii). PI

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds$$

$$\int_{-a}^{a} dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\ \frac{\sin as}{s}\right)^2 ds$$

$$(x)_{-a}^{a} = \frac{2}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds$$

$$2a\frac{\pi}{2} = 2\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds$$

$$\int_0^\infty \left(\frac{\sin as}{s}\right)^2 ds = \frac{\pi}{2}a$$

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{2} \qquad (\because a=1)$$

$$\Rightarrow \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

Transforms of simple functions :

1. Find the fourier transform of

$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ and hence prove that

$$\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$$

Soln :

Given $f(x) = \begin{cases} 1-x^2, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\,(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[2\int_0^1 (1-x^2)\cos sx\, dx + i(0)\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + \cancel{(-2)} \right.$$
$$\left. (-2)\left(-\frac{\sin sx}{s^3}\right) - 0\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\frac{\sin sx}{s} - \frac{2x\cos sx}{s^2} + \frac{2\sin sx}{s^3}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3}\right) - 0\right] \cancel{2\sin}$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right]$$

By using IFT,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] (\cos sx - i \sin sx) \, ds$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx \, ds \right]$$

$$= \frac{4}{\pi} \left[\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds - i(0) \right]$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx \, ds = \frac{\pi}{4} f(x)$$

$\because f(x) = 1 - x^2$

$f(1/2) = 1 - 1/4 = 3/4$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos s/2 \, ds = \frac{3\pi}{16}$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)$$