Problems on Self reciprocal function:

1. Show that the function $e^{-x^2/2}$ is self-reciprocal under fourier transform.

Soln.:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2 - 2isx + (is)^2 - (is)^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-is)^2 + s^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} e^{-\frac{s^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-is}{\sqrt{2}}\right)^2}\, dx$$

Put $t = \frac{x-is}{\sqrt{2}}$

$dt = \frac{dx}{\sqrt{2}}$

$dx = \sqrt{2}\, dt$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-s^2/2} \sqrt{\pi} \qquad \because \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$$

$$F(s) = e^{-s^2/2}$$

~~Hence~~

The Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self reciprocal under F.T.

~~2] H.W Find the Fourier transform of $f(x) = e^{-x^2}$~~

2]. Find the Fourier transform of $f(x) = e^{-a^2x^2}$.

Soln. :

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 + isx)}\, dx$$

$2ax\,b = is$

$b = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}\, dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2}\, dx$$

Put $t = ax - \frac{is}{2a}$

$dt = a\, dx$

$dx = \frac{dt}{a}$

$x = -\infty \Rightarrow t = -\infty$

$x = \infty \Rightarrow t = \infty$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a}$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi}$$

$$= e^{-s^2/4a^2} \cdot \frac{1}{a\sqrt{2}}$$