UNIT 4

COMPLEX INTEGRATION

CAUCHY'S INTEGRAL THEOREM:

If a function f(z) is analytic and its derivative $f'(z)$ is continuous with all points inside and on a simple closed curve C then $\int_c f(z)dz = 0$

CAUCHY'S INTEGRAL FORMULA:

If f(z) is analytic inside and on a simple closed curve C and 'a' be any point inside C then $\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$,where the integration being taken in the anticlockwise direction around C.

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES:

If f(z) is analytic inside and on a simple closed curve C and let 'a' be any point inside C then $\int_C \frac{f(z)}{(z-a)^2}dz = \frac{2\pi i f'(a)}{1!}$

and $\int_C \frac{f(z)}{(z-a)^3}dz = \frac{2\pi i f''(a)}{2!}$

In general,

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = 2\pi i f^{(n)}(a)$$

Solved Problems:

1. Evaluate $\int_C \frac{Cos\,\pi z}{z-1}dz$ where the circle C is |z|=2

Solution:

$$f(z) = \cos \pi z$$
$$z - 1 = 0$$
$$z = 1$$
$$Given\ condition : |z| = 2$$
$$i.e., |1| < 1 \text{ which lies inside C.}$$

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\therefore \int_C \frac{Cos\, \pi z}{z-1} dz = 2\pi i f(1) = -2\pi i$$

2.Evaluate $\int_C \frac{dz}{z+2}$ where C is $|z|=1$

Solution:

$$f(z) = 1$$
$$z + 2 = 0$$
$$z = -2$$
$$Given\ condition : |z| = 1$$
$$i.e., |-2| = 2 > 1 \text{ which lies outside C.}$$

Therefore by Cauchy's integral formula $\int_C \frac{dz}{z+2} = 0$

3.Evaluate $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z+\frac{1}{2}|=1$

Solution:

$$f(z) = e^z$$
$$z+1=0$$
$$z=-1$$
$$Given\,condition : \left|z+\frac{1}{2}\right| = 1$$
$$i.e., \left|-1+\frac{1}{2}\right| = \frac{1}{2} < 1 \text{ which lies inside C.}$$

Therefore by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\frac{e^z}{z+1} dz = 2\pi i f(-1)$$
$$= 2\pi i e^{-1}$$
$$= \frac{2\pi i}{e}$$

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4. Using Cauchy's integral formula $\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is circle$|z|=1$ or $x^2+y^2=9$

Solution:

Consider

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \qquad \text{.........................(1)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2)}{z-1} + \frac{B(z-1)}{z-2}$$

$1= A(z-2)+B(z-1)$

Substituting z=2,

$1=0+B \Rightarrow B=1$

Substituting z=1,

1=A(-1+0)$\Rightarrow$ A=-1

Substituting A=-1, B=1 in (1)

$$\frac{1}{(z-1)(z-2)}=\frac{-1}{z-1}+\frac{1}{z-2}$$

Multiply both sides by $\sin\pi z^2+\cos\pi z^2$,we get

$$\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}=\frac{-(\sin\pi z^2+\cos\pi z^2)}{z-1}+\frac{(\sin\pi z^2+\cos\pi z^2)}{z-2}$$

$$\int\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-\int\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)}dz+\int\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}dz$$

f(z)= $\sin\pi z^2+\cos\pi z^2$

$$f(z)=\sin\pi z^2+\cos\pi z^2$$

$$z-1=0 \quad and \quad z-2=0$$

$$z=1 \qquad\qquad z=2$$

$Given\, condition: |z|=3$

$i.e., |1|<3 \;\; and \;\; |2|<3$ which lies inside C.

$$\therefore \int\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)(z-2)}dz=-2\pi i\, f(1)+2\pi i\, f(2)$$

$$=-2\pi i(-1)+2\pi i\;(2)$$

$$=4\pi i$$ (since f(1)=-1 and f(2)=1)

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5. Evaluate $\int\frac{e^z}{(z-1)^3}dz$ where C is the circle |z-1|=$\frac{3}{2}$

Solution:

Substituting z=1,

1=A(-1+0) $\Rightarrow$ A=-1

Substituting A=-1, B=1 in (1)

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Multiply both sides by $\sin \pi z^2 + \cos \pi z^2$,we get

$$\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}$$

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

f(z)= $\sin \pi z^2 + \cos \pi z^2$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$z-1=0 \quad and \quad z-2=0$$

$$z=1 \qquad\qquad z=2$$

Given condition : $|z| = 3$

$i.e., |1| < 3 \ \ and \ \ |2| < 3$ which lies inside C.

$$\therefore \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i\, f(1) + 2\pi i\, f(2)$$

$$= -2\pi i(-1) + 2\pi i\ (2)$$

$$= 4\pi i$$ (since f(1)=-1 and f(2)=1)

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5. Evaluate $\int \frac{e^z}{(z-1)^3} dz$ where C is the circle |z-1|= $\frac{3}{2}$

Solution:

$f(z) = e^z$

$z - 1 = 0$

$z = 1$,

$Given\, condition : |z-1| = \frac{3}{2}$

$i.e., |1-1| = 0 < \frac{3}{2}$ which lies inside C.

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i f''(a)}{2!}$$

Therefore $\int_C \frac{e^z}{(z-1)^3} dz = \frac{2\pi i f''(1)}{2!} = \frac{2\pi i e}{2} = \pi i e$