



cauchy's integral theorem :-

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C , then $\int_C f(z) dz = 0$.

cauchy's integral formula

If $f(z)$ is analytic inside and on simple closed curve C and a be any point inside C then $\int_C \frac{f(z) dz}{z-a} = 2\pi i f(a)$, where the integration being taken around the anti-clockwise direction.

cauchy's integral formula for derivatives :-

If the function $f(z)$ is analytic inside and on a simple closed curve C , then and a be any point inside C , then,

$$\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3} dz = 2\pi i f''(a)$$

In general,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i f^{(n)}(a)$$

If the point a lies outside the circle or region, then $\int_C \frac{f(z)}{(z-a)} dz = 0$

Problem :-

1) Evaluate $\int_C \frac{e^{\cos z}}{z-1} dz$ using cauchy's integral



formula, where c is the circle $|z|=2$
Soln: Here $z=1$, $|1|=1 < 2$ which lies inside c .

By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_c \frac{\cos \pi z}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i (-1)$$

$$\int_c \frac{\cos \pi z}{z-1} dz = -2\pi i$$

2) Evaluate $\int_c \frac{dz}{(z-3)^2}$ where c is the circle $|z|=1$

Soln:

$$c, |z|=1$$

$$c: |z|=1$$

Here $z=3$, $|3|=3 > 1$, which lies outside c ,

By Cauchy's integral formula,

$$\int_c \frac{dz}{(z-3)^2} = 0$$

3) Evaluate $\int_c \frac{e^z}{z+1} dz$, where c is the circle $|z+\frac{1}{2}|=1$

Soln: Given: $c: |z+\frac{1}{2}|=1$

$$z+1=0$$

Here $z=-1$, $|-1+\frac{1}{2}|=|-1/2|=1/2 < 1$, which lies inside c .

By Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\int_c \frac{e^z}{z-(-1)} dz = 2\pi i f(-1)$$



UNIT-IV COMPLEX INTEGRATION

CAUCHY'S INTEGRAL FORMULA

$$\int_C \frac{e^z}{z+1} dz = 2\pi i \left(\frac{1}{e} \right)$$

$$\int_C \frac{e^z}{z+1} dz = \frac{2\pi i}{e}$$

Here, $f(z) = e^z$

$$f(-1) = e^{-1}$$

$$= \frac{1}{e}$$

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$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz, \quad c: |z| = 3$$

Soln:-

$$c: |z| = 3$$

$$(z-1)(z-2) = 0$$

Here $z=1, |1|=1 < 3$, which lies inside c .

$z=2, |2|=2 < 3$ which lies inside c

By using partial fraction method,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1) \rightarrow$$

Put $z=1,$

$$1 = A(1-2) + B(1-1)$$

$$1 = -A \rightarrow A = -1$$

put $z=2$

$$\Rightarrow 1 = A(2-2) + B(2-1)$$

$$1 = A(0) + B(1)$$

$$1 = B$$

$$B = 1$$

Now,

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$f(z) = \sin \pi + \cos \pi$$

$$f(z) = 0 - 1 = -1$$

$$f(2) = \sin \pi (2)^2 + \cos \pi (2)^2$$

$$f(2) = 0 + 1 = 1$$

$$\sin n\pi = 0$$

$$\cos \pi, 3\pi, 5\pi = -1$$

$$\cos 2\pi, 4\pi, 6\pi = 1$$

$$\cos n\pi = (-1)^n$$



$$\Rightarrow \int_c \frac{-\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

$$\Rightarrow -2\pi i f(1) + 2\pi i f(2)$$

$$\Rightarrow -2\pi i (-1) + 2\pi i (1)$$

$$\Rightarrow 2\pi i + 2\pi i$$

$$\Rightarrow 4\pi i$$

5) Evaluate $\int_c \frac{dz}{z^2 - 2z}$, where c is the circle $|z| = 1$

Soln:- $\int_c \frac{dz}{z^2 - 2z} = \int_c \frac{dz}{z(z-2)}$

Given :- $c: |z| = 1$
 $z(z-2) = 0$

Here $z=0$, $|0| = 0 < 1$, which lies inside c .
 $z=2$, $|2| = 2 > 1$, which lies outside c .

$$\int_c \frac{dz}{z(z-2)} = \int_c \frac{1}{z} dz \rightarrow \textcircled{1}$$

By Cauchy's ^{integral} formula

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\textcircled{1} \Rightarrow \int_c \frac{dz}{z^2 - 2z} = 2\pi i f(0)$$

$$= 2\pi i f(-1/2) = -\pi i$$

6) Evaluate $\int_c \frac{z dz}{(z-1)(z-2)^2}$ where c is the circle

$$|z-2| = 1/2$$

Given :- $c: |z-2| = 1/2$

$$(z-1)(z-2)^2 = 0$$

Here $z=1$, $|1-2| = |1-2| = 1 > 1/2$ which lies outside c



$z=2$, $|2-2|=0 < 1/2$, which lies inside C

$$\int_C \frac{z dz}{(z-1)(z-2)^2} = \int_C \frac{z}{(z-1)^2} dz$$

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\textcircled{1} \Rightarrow \int_C \frac{z dz}{(z-1)(z-2)^2} = 2\pi i f'(1/2)$$

$$= 2\pi i (-1)$$

$$= -2\pi i$$