Taylor's Series:-

A function $f(z)$ is analytic inside a circle 'c' with centre at 'a' can be expressed in the series $f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \dots$ which is convergent at a ~~any~~ every point inside 'c'

Note:-

If 'a' = 0, then

$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \dots$, which is called as maclaurin series.

1) Expand $f(z) = \log(1+z)$ as taylor's series about $z=0$

Soln:-

$f(z) = \log(1+z)$ | $f(0) = \log(1+0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z}$ | $f'(0) = \frac{1}{1+0} = 1$

$f''(z) = \frac{-1}{(1+z)^2}$ | $f''(0) = \frac{-1}{(1+0)^2} = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ | $f'''(0) = \frac{2}{(1+0)^3} = 2$

By Taylor's series

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a)$$

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0)$$

$$f(z) = 0 + z(1) + \frac{z^2}{2}(-1) + \frac{z^3}{6}(2) + \dots$$

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

3) Expand $f(z) = e^z$ as taylor series about $z=0$

Soln:-

$f(z) = e^z$	$f(0) = e^0 = 1$
$f'(z) = e^z$	$f'(0) = e^0 = 1$
$f''(z) = e^z$	$f''(0) = e^0 = 1$
$f'''(z) = e^z$	$f'''(0) = e^0 = 1$

By taylor's expansion

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0)$$

$$f(z) = 1 + \frac{z}{1}(1) + \frac{z^2}{2}(1) + \frac{z^3}{6}(1) + \dots$$

$$f(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \dots$$

$$f(z) = 1 + z + \frac{z^2}{2} + \frac{z^2}{3} + \dots$$

4) Expand $f(z) = \cos z$ as taylor series at $z = \pi/3$

Soln:-

$f(z) = \cos z$	$f(\pi/3) = \cos \pi/3 = 1/2$
$f'(z) = -\sin z$	$f'(\pi/3) = -\sin \pi/3 = -\sqrt{3}/2$
$f''(z) = -\cos z$	$f''(\pi/3) = -\cos \pi/3 = -1/2$
$f'''(z) = +\sin z$	$f'''(\pi/3) = \sin \pi/3 = \sqrt{3}/2$

By Taylor's series,

$$f(z) = f(a) + \frac{z-a}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

$$f(z) = f(\pi/3) + \frac{z-\pi/3}{1} f'(\pi/3) + \frac{(z-\pi/3)^2}{2} f''(\pi/3) + \frac{(z-\pi/3)^3}{2} f'''(\pi/3) + \cdots$$

$$f(z) = 1/2 + (z-\pi/3)(-\sqrt{3}/2) + \left(\frac{(z-\pi/3)^2}{2}\right)(-1/2) + \left(\frac{z-\pi/3}{6}\right)^3 \left(\sqrt{3}/2\right)$$