Cauchy's Residue theorem

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except at finite number of singular point $z_1, z_2, \dots z_n$ inside C. Then

$$\int f(z)\,dz = 2\pi i\,[\text{sum of residues of } z_1, z_2, \dots z_n]$$

1) Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz$ where c is the circle, modulus $|z| = 3$ using cauchy's residue theorem.

Soln:- Let

$$f(z) = \frac{e^z}{(z+2)(z+1)^2}$$

The poles of $f(z)$ are, $(z+2)(z+1)^2 = 0$

Here, $z=-2$ is a pole of order 1

$z=-1$ is a pole of order 2

Given $|z|=3$

$|z| = |-2| = 2 < 3$, lies inside c

$|z| = |-1| = 1 < 3$, lies inside c.

$z = -2$ is a pole of order 1

$$\left[\text{Res } f(z)\right]_{z=a} = \lim_{z \to a} (z-a) f(z)$$

$$\left[\text{Res } f(z)\right]_{z=-2} = \lim_{z \to -2} (z-(-2)) \frac{e^z}{(z+2)(z+1)^2}$$

$$= \lim_{z \to -2} (z+2) \frac{e^z}{(z+2)(z+1)^2}$$

$$= \lim_{z \to -2} \frac{e^z}{(z+1)^2}$$

$$= \frac{e^{-2}}{(-2+1)^2} = \frac{e^{-2}}{(-1)^2}$$

$$\left[\text{Res } f(z)\right]_{z=-2} = e^{-2}$$

$z = -1$ is a pole of order 2 $(m = 2)$

$$\left[\text{Res } f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

$$\left[\text{Res } f(z)\right]_{z=-1} = \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d^{2-1}}{dz^{2-1}} \left[(z-(-1))^2 \frac{e^z}{(z+2)(z+1)^2} \right]$$

$$\left[\text{Res } f(z)\right]_{z=-1} = \lim_{z \to -1} \frac{d}{dz} \left[\frac{e^z}{z+2}\right]$$

$$\left[\text{Res } f(z)\right]_{z=-1} = \lim_{z \to -1} \left[\frac{(z+2)e^z - e^z(1)}{(z+2)^2} \right]$$

$$\left[\text{Res } f(z)\right]_{z=-1} = \lim_{z = -1} \frac{(z+2)e^z - e^z}{(z+2)^2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z=-1} \frac{ze^{z}+2e^{z}-e^{z}}{(z+2)^2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z\to-1} \frac{ze^{z}+e^{z}}{(z+2)^2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{(-1)e^{-1}+e^{-1}}{(-1+2)^2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{-e^{-1}+e^{-1}}{1}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = 0$$

By cauchy's Residue theorem,

$$\int_c \frac{e^z}{(z+2)(z+1)^2}\,dz = 2\pi i\left[e^{-2}\right]$$

2). Evaluate $\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz$, where c is the circle $|z| = 3/2$ using cauchy's residue theorem.

<u>Soln:-</u> let $\frac{4-3z}{z(z-1)(z-2)}$

The poles of $f(z)$ are $z(z-1)(z-2)=0$

$z=0$, is a pole of order 1

$z=1$, is a pole of order 1

$z=2$, is a pole of order 1

Given $|z| = 3/2 = 1.5$

$|z| = |0| = 0 < 3/2$, lies inside c

$|z| = |1| = 1 < 3/2$, lies inside c

$|z| = |2| = 2 < 3/2$, lies inside c

Here $z=0$ is a pole of order 1

$$[\text{Res } f(z)]_{z=0} = \lim_{z\to 0} (z-0)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{4-3z}{(z-1)(z-2)} = \frac{4-0}{-1(-2)} = \frac{4}{2} = 2$$

$$[\text{Res } f(z)]_{z=0} = 2$$

Here $z=1$ is a pole of order 1

$$[\text{Res } f(z)]_{z=1} = \lim_{z\to 1} (z-1)\frac{4-3z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{4-3z}{z(z-2)}$$

$$= \frac{4-3(1)}{1(1-2)} = \frac{4-3}{1-2} = \frac{1}{-1}$$

$$[\text{Res } f(z)]_{z=1} = -1$$

3) Evaluate $\int_c \frac{z}{(z-1)^2(z+1)}\,dz$, where c is the circle is $|z| = 1/2$, $|z| = 2$

Soln:- $f(z) = \dfrac{z}{(z-1)^2(z+1)}$

The poles of $f(z)$ are $(z-1)^2(z+1) = 0$

$z = 1$, is a pole of order 2

$z = -1$ is a pole of order 1.

Given is $|z| = 1/2$

$|z| = |1| = 1 > 1/2$, lies outside c

$|z| = |-1| = +1 > 1/2$, lies outside c

$[\text{Res } f(z)]_{z=1} = 0$ and $[\text{Res } f(z)]_{z=-1} = 0$

By cauchy's Residue theorem,

$$\int_c \frac{z}{(z-1)^2(z+1)} dz = 2\pi i(0) = 0$$

ii) Given $|z| = 2$

$|z| = |1| = 1 < 2$, lies inside c

$|z| = |-1| = 1 < 2$, lies inside c

Here $z = 1$ is a pole of order 2 $(m=2)$

$$[\text{Res } f(z)]_{z=1} = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} \left[(z-1)^2 \frac{z}{(z-1)^2(z+1)} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{z}{z+1} \right]$$

$$= \lim_{z \to 1} \left[\frac{(z+1)(1) - z(1)}{(z+1)^2} \right]$$

$$= \lim_{z \to 1} \left[\frac{z+1-z}{(z+1)^2} \right]$$

$$= \lim_{z \to 1} \left[\frac{1}{(z+1)^2} \right]$$

$$= \frac{1}{(1+1)^2}$$

$$[\text{Res } f(z)]_{z=1} = 1/4$$

$z = -1$ is a pole of order 1

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z \to -1} (z+1) \frac{z}{(z-1)^2(z+1)}$$

$$= \lim_{z \to -1} \frac{z}{(z-1)^2}$$

$$= \frac{-1}{(-1-1)^2}$$

$$= -1/4$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = -1/4$$

By cauchy's Residue theorem,

$$\int_C \frac{z}{(z-1)^2(z+1)} dz = 2\pi i \left(\frac{1}{4} - \frac{1}{4}\right) = 2\pi i[0] = 0$$