Laurent's series:

Let C_1 and C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let f(z) be analytic on C_1 and C_2 and in the annular region R between them. Then, for any point z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{(z-a)^{n+1}} dz$$

and $$b_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

The integrals being taken in the anticlockwise direction.

Note:

In Laurent's series of f(z) , the terms containing positive powers is called regular part and the terms containing negative powers is called principle part.

Solved Problems:

1.Expand $f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$ in Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$ (iii) $2 < |z| < 3$ (iv) $1 < |z+1| < 3$

Solution:

Consider

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2) \text{ ....(1)}$$

Put z=2 in (1),

$$7(2)-2 = A(0) + B(2)(2+1) + C(0)$$

$$\therefore B = 2$$

Put z=-1 in (1),

$$7(-1)-2 = A(0) + B(0) + C(-1)(-1-2)$$

$$\therefore C = -3$$

Put z=0 in (1),

$$7(0)-2 = A(0-2) + B(0) + C(0)$$

$$\therefore A = 1$$

$$\therefore \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} + \frac{-3}{z+1}$$

(i) $|z| < 2$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{2\left(\frac{z}{2}-1\right)} - \frac{3}{z+1}$$

$$= \frac{1}{z} - \left(1-\frac{z}{2}\right)^{-1} - 3(1+z)^{-1}$$

$$= \frac{1}{z} - \left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2 + ...\right) - 3(1-z+z^2-....)$$

(ii) $|z| > 3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^{2}+....\right) - \frac{3}{z}\left(1-\frac{1}{z}+\frac{1}{z^{2}}-....\right)$$

(iii) $2 < |z| < 3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z\left(1-\frac{2}{z}\right)} - \frac{3}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z} + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^{2}+....\right) - \frac{3}{z}\left(1-\frac{1}{z}+\frac{1}{z^{2}}-....\right)$$

(iv) $1 < |z+1| < 3$

$$\frac{7z-2}{z(z-2)(z+1)}=\frac{1}{z}+\frac{2}{z-2}+\frac{-3}{z+1} \quad \ldots\ldots(2)$$

Let $t=z+1$.

$$\Rightarrow z=t-1$$

Given condition: $1<|z+1|<3 \Rightarrow 1<|t|<3$

$$\frac{7z-2}{z(z-2)(z+1)}=\frac{1}{t-1}+\frac{2}{t-3}-\frac{3}{t}$$

$$=\frac{1}{t\left(1-\frac{1}{t}\right)}+\frac{2}{3\left(\frac{t}{3}-1\right)}-\frac{3}{t}$$

$$=\frac{1}{t}\left(1-\frac{1}{t}\right)^{-1}-\frac{2}{3}\left(1-\frac{t}{3}\right)^{-1}-\frac{3}{t}$$

$$=\frac{1}{t}\left(1+\frac{1}{t}+\frac{1}{t^2}+\ldots\right)-\frac{2}{3}\left(1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\ldots\right)-\frac{3}{t}$$

$$=\frac{1}{z+1}\left(1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\ldots\right)-\frac{2}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\ldots\right)-\frac{3}{z+1}$$

2.Expand $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ in a Laurent's series if (i) $2<|z|<3$ (ii) $|z|>3$

Solution:

Simplify $f(z)=\dfrac{z^2-1}{(z+2)(z+3)}$ using long division, we get

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=1+\frac{-5z-7}{z^2+5z+6}$$

$$=1-\frac{5z+7}{z^2+5z+6}$$

Consider $\dfrac{5z+7}{z^2+5z+6}=\dfrac{5z+7}{(z+2)(z+3)}=\dfrac{A}{z+2}+\dfrac{B}{z+3}$

$$5z+7=A(z+3)+B(z+2)\ \ldots\ldots(1)$$

Put z=-3 in (1) ,

$$5(-3)+7=A(-3+3)+B(-3+2)$$

$$\therefore B=-8$$

Put z=-2 in (1) ,

$$5(-2)+7=A(-2+3)+B(-2+2)$$

$$\therefore A=3$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}\ldots\ldots(2)$$

(i) $2<|z|<3$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{8}{3}\left(1-\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} + \ldots\ldots\right) + \frac{8}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \ldots\ldots\right)$$

(ii) $|z| > 3$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} + \ldots\ldots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{9}{z^2} + \ldots\ldots\right)$$