NUMERICAL INTEGRATION

(6)

TRAPEZOIDAL RULE:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(\text{sum of the first and the last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

where $h = \dfrac{b-a}{n}$

SIMPSON'S ONE-THIRD RULE:

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$$

$$= \frac{h}{3}\left\{[\text{Sum of the first and last ordinates}] + 2[\text{Sum of the remaining odd ordinates}] + 4[\text{Sum of the even ordinates}]\right\}$$

Note:

1. Even though y_2 has suffix even, it is the third ordinate (odd).
2. This formula can be applied only if the number of intervals is even. (or) the number of ordinates is odd.

SIMPSON'S THREE-EIGHTH RULE:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_n)\right]$$

which is applicable only when n is a multiple of 3.

(7)

(1) Evaluate $\int_0^1 \frac{dx}{1+x^2}$, using Trapezoidal rule with $h = 0.2$. Hence Obtain an approximate value of π.

Solution:

x :	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$:	1	0.9615	0.8621	0.7353	0.6098	0.5

By Trapezoidal rule,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1+0.5)+2(0.9615+0.8621+0.7353+0.6098)\right]$$

$$= \underline{0.7837}$$

To find the value of π:

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

$$\therefore \frac{\pi}{4} = 0.7837 \Rightarrow \boxed{\pi = 3.1348} \text{ approximately.}$$

(2) Evaluate $I = \int_0^6 \frac{1}{1+x}\,dx$ using (i) Trapezoidal rule (ii) Simpson's rule. Also, Check up by direct integration.

Solution:

Take the number of intervals n as 6.

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1.$$

x :	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$:	1	0.5	0.3333	0.25	0.2	0.1667	0.1429

(i) By Trapezoidal rule,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left\{ (y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1}) \right\}$$

$$= \frac{1}{2}\left\{ (1+0.1429) + 2(0.5+0.3333+0.25+0.2+0.1667) \right\}$$

$$= \underline{2.02145}$$

(ii) By Simpson's one-third rule,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left\{ (y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots) \right]$$

$$= \frac{1}{3}\left\{ (1+0.1429) + 2(0.3333+0.2) + 4(0.5+0.25+0.1667) \right]$$

$$= \cancel{2.7006}$$

$$= \underline{1.9588}$$

(iii) By Simpson's 3-8th rule,

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left\{ (y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+y_9+\cdots) \right\}$$

$$= \frac{3}{8}\left\{ (1+0.1429) + 3(0.5+0.3333+0.2+0.1667) + 2(0.25) \right\}$$

$$= \underline{1.9661}$$

(iv) By actual integration,

$$\int_0^6 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^6 = \log 7 - \log 1 = \log_e 7 - 0$$

$\ln 7$

$$= \underline{1.9459}$$

(8)

(3) The velocity v of a particle at distance 's' from a point on its linear path is given in the following table:

S (m) :	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
v (m/sec) :	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 meters, using Simpson's one-third rule.

Solution:

$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} \Rightarrow t = \int \frac{1}{v}\, ds$$

$$\therefore \text{Time required} = \int_0^{20} \frac{1}{v}\, ds$$

S :	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
$y = \frac{1}{v}$:	0.0625	0.0526	0.0476	0.0455	0.05	0.0588	0.0769	0.0909	0.1111
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By Simpson's one-third formula,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_0^{20} \frac{1}{v}\,ds = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\cdots) + 4(y_1+y_3+\cdots)\right]$$

$$= \frac{2.5}{3}\left[(0.0625+0.1111) + 2(0.0476+0.05+0.0769) + 4(0.0526+0.0455+0.0588+0.0909)\right]$$

$$= 1.2615 \text{ seconds}$$

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(3) Compute $\int_{4}^{5.2} \log_e x \, dx$, using Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule.

Solution:

Here $f(x) = \log_e x$

$$h = \frac{b-a}{n} = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$y = \log_e x$	1.386	1.435	1.482	1.526	1.569	1.609	1.649

By Simpson's $1/3^{rd}$ rule,

$$I = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\ldots) + 4(y_1+y_3+\cdots)\right]$$

$$= \frac{0.2}{3}\left[(1.386+1.649) + 2(1.482+1.569) + 4(1.435+1.526+1.609)\right]$$

$I = 1.828$

By Simpson's $3/8^{th}$ rule,

$$I = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+\ldots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots)\right]$$

$$= \frac{3 \times 0.2}{8}\left[(1.386+1.649) + 3(1.435+1.482+1.569+1.609) + 2(1.526 + \cancel{1.649})\right]$$

$I = 1.828$

(4) S is the specific heat of a body at temperature $\theta^{\circ}C$. Find the total heat required to raise the temperature of the body of weight 1 gram from $0^{\circ}C$ to $12^{\circ}C$, using the following data of values and Simpson's $3/8^{th}$ rule.

θ	0	2	4	6	8	10	12
S	1.00664	1.00543	1.00435	1.00331	1.00233	1.00149	1.00078

Solution : The total heat required is given by,

$$H = \int_0^{12} S \, d\theta \quad \text{and} \quad h = \frac{b-a}{n} = \frac{12}{6} = 2$$

$$= \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_4) + 3(y_2+y_5) + 2y_3\right]$$

$$= \frac{3\times 2}{8}\left[(1.00664 + 1.00078) + 3(1.00543 + 1.00233) + 3(1.00435 + 1.00149) + 2\times 1.0031\right]$$

$$H = 12.04113$$

(5) Dividing the range into 10 equal parts, find the approximate value of $\int_0^{\pi} \sin x \, dx$ by trap rule.

Soln: $h = \frac{\pi - 0}{10} = \pi/10$

x :	0	$\pi/10$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
y :	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090	0

$$I = 1.9843$$

(6) Calculate $\int_0^{\pi} \sin^3 x \, dx$ taking $h = \pi/6$

$$I = 1.305$$

(7) Apply Simpson's rule to find the value of $\int_0^{2} \frac{dx}{1+x^3}$ dividing into 4 equal parts.

$$I = 1.096.$$