SNS COLLEGE OF TECHNOLOGY

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Coimbatore-35

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DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

19ECB212 – DIGITAL SIGNAL PROCESSING

II YEAR/ IV SEMESTER

UNIT 5 – DSP APPLICATIONS

TOPIC – SPECTRUM OF MULTIRATE DSP

MULTIRATE DSP

- The processing of a discrete time signal at different sampling rates in different parts of a system is called **multirate DSP**
- The discrete time systems that employ sampling rate conversion while processing the discrete time signals are called **multirate DSP Systems**
- **Downsampling or decimation** is the process of reducing the sampling rate by an integer factor D
- **Upsampling or interpolation** is the process of increasing the sampling rate by an integer factor I

SPECTRUM OF DOWNSAMPLER

- Let x(n) be an input signal to the downsampler and y(n) be the output signal
- Let x'(nD) be the downsampled version of x(n) by an integer factor D

$$y(n) = x'(nD)$$

- Consider a unit pulse train defined as

$$p(n) = 1 \text{ ; for } n = 0, \pm D, \pm 2D, \pm 3D, \ldots\ldots$$

$$= 0 \text{ ; otherwise}$$

- Consider the product of x(n) and p(n)

$$x(n)\,p(n) = x(n) \text{ ; for } n = 0, \pm D, \pm 2D, \ldots\ldots$$

$$= 0 \text{ ; otherwise}$$

- Now x’(n) is the signal obtained after removing all zeros from x(n) p(n)

$$x'(n) = x(n)\,p(n) \text{ ; for } n = 0, \pm D, \pm 2D, \ldots\ldots$$

Z TRANSFORM

$$Y(z) = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x'(nD)\, z^{-n}$$

On substituting, y(n) = x′(nD), from equation

$$= \sum_{m=-\infty}^{+\infty} x'(m)\, z^{-\frac{m}{D}}$$

Let, $m = nD \Rightarrow n = \frac{m}{D}$

When $n = -\infty$, $m = -\infty$

When $n = +\infty$, $m = +\infty$

$$= \sum_{n=-\infty}^{+\infty} x'(n)\, z^{-\frac{n}{D}}$$

Let, $m = n$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, p(n)\, z^{-\frac{n}{D}}$$

On substituting, $x'(n) = x(n)\, p(n)$, from equation

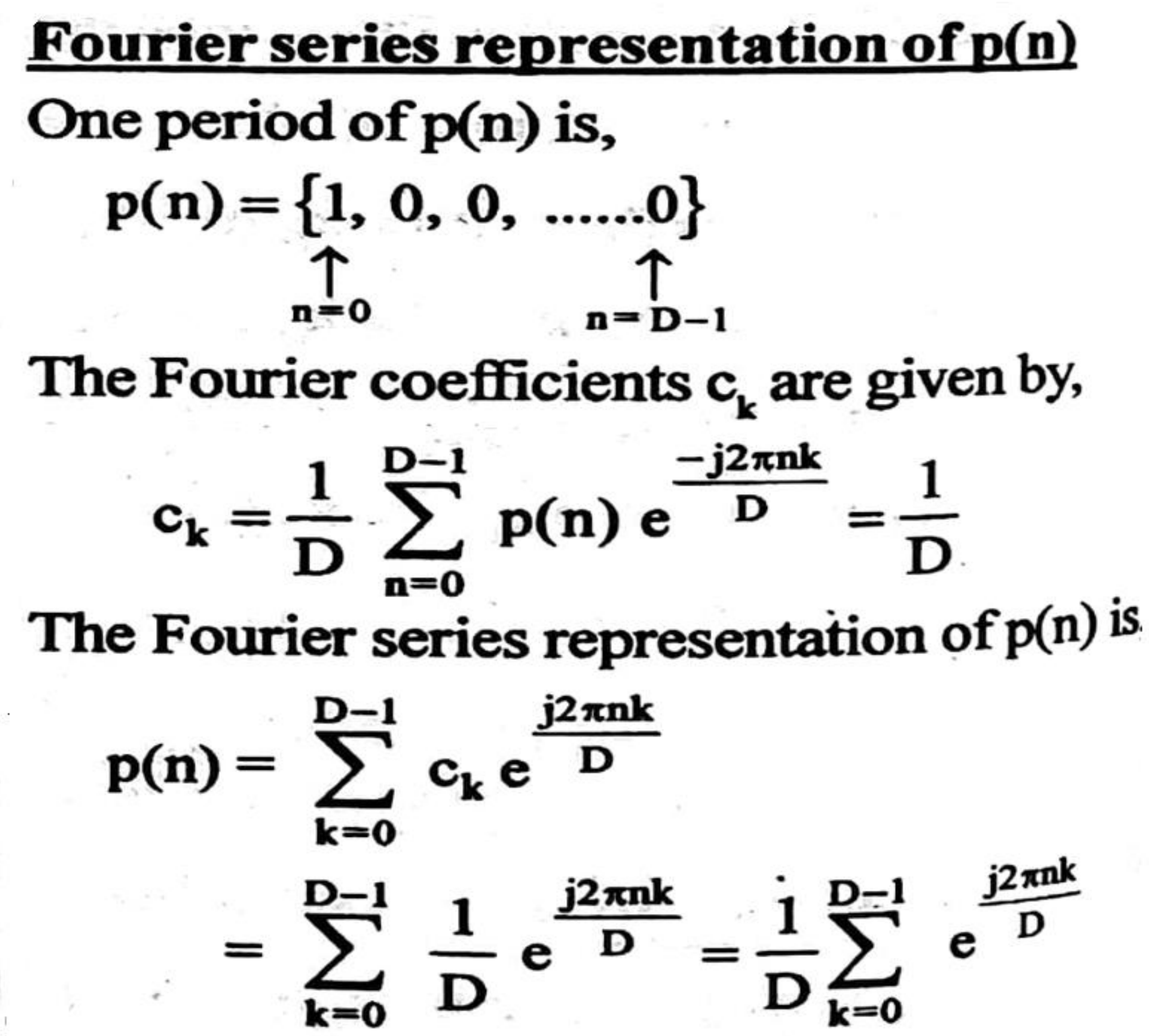
$$= \sum_{n=-\infty}^{+\infty} x(n) \left[\frac{1}{D} \sum_{k=0}^{D-1} e^{\frac{j2\pi kn}{D}} \right] z^{-\frac{n}{D}}$$

Replacing p(n) by its Fourier series representation.

$$= \frac{1}{D} \sum_{k=0}^{D-1} \left[\sum_{n=-\infty}^{+\infty} x(n)\, e^{\frac{j2\pi kn}{D}}\, z^{-\frac{n}{D}} \right]$$

$$= \frac{1}{D} \sum_{k=0}^{D-1} \left[\sum_{n=-\infty}^{+\infty} x(n) \left[e^{\frac{-j2\pi k}{D}}\, z^{\frac{1}{D}} \right]^{-n} \right]$$

SPECTRUM OF DOWNSAMPLER



Fourier series representation of p(n)

One period of p(n) is,

$$p(n) = \{\underset{\underset{n=0}{\uparrow}}{1},\ 0,\ 0,\ \ldots\ldots\underset{\underset{n=D-1}{\uparrow}}{0}\}$$

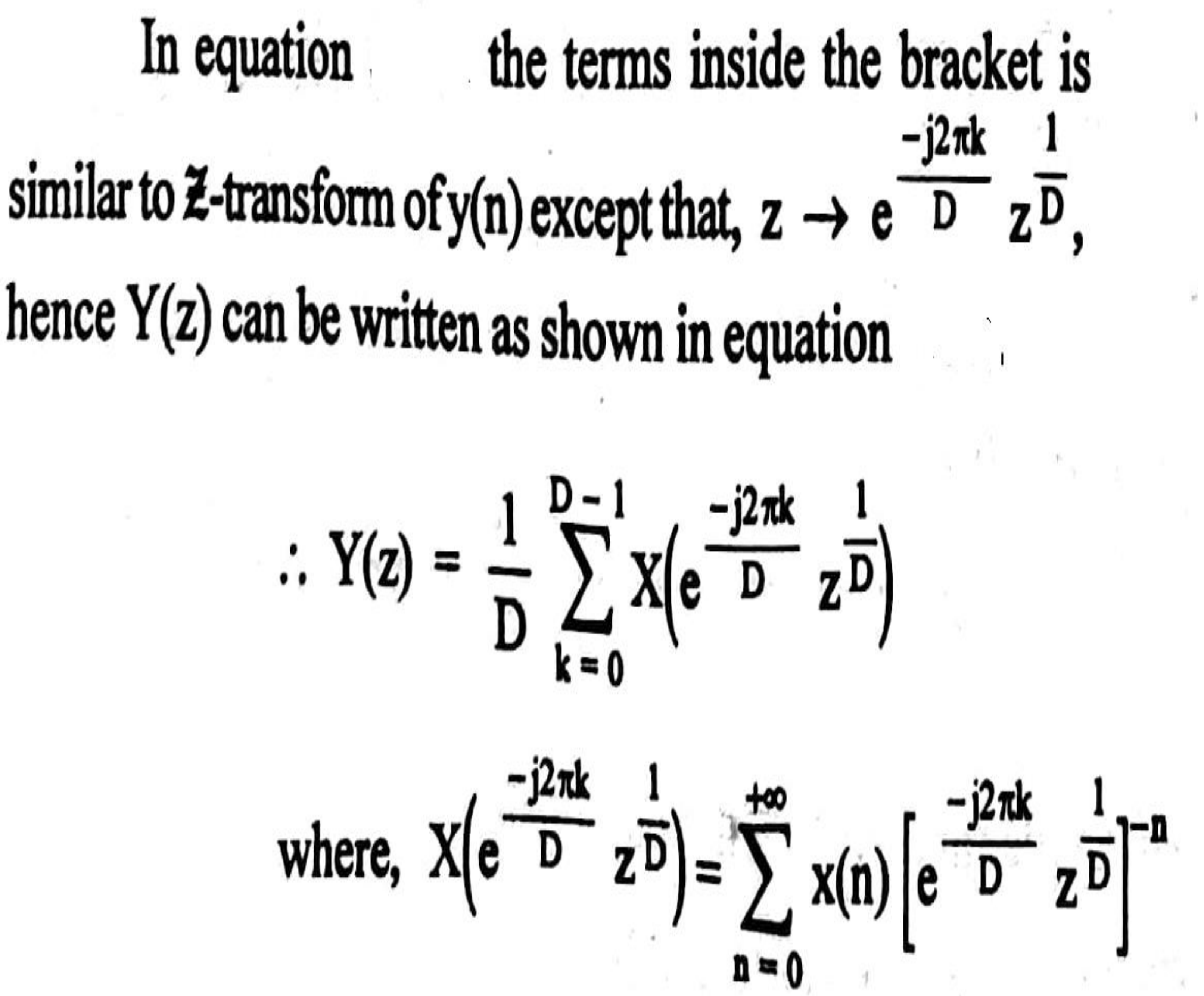
The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{D}\sum_{n=0}^{D-1} p(n)\, e^{\frac{-j2\pi nk}{D}} = \frac{1}{D}$$

The Fourier series representation of p(n) is

$$p(n) = \sum_{k=0}^{D-1} c_k\, e^{\frac{j2\pi nk}{D}}$$

$$= \sum_{k=0}^{D-1} \frac{1}{D}\, e^{\frac{j2\pi nk}{D}} = \frac{1}{D}\sum_{k=0}^{D-1} e^{\frac{j2\pi nk}{D}}$$



In equation the terms inside the bracket is similar to Z-transform of y(n) except that, $z \to e^{\frac{-j2\pi k}{D}} z^{\frac{1}{D}}$, hence Y(z) can be written as shown in equation

$$\therefore Y(z) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{\frac{-j2\pi k}{D}} z^{\frac{1}{D}}\right)$$

$$\text{where, } X\left(e^{\frac{-j2\pi k}{D}} z^{\frac{1}{D}}\right) = \sum_{n=0}^{+\infty} x(n)\left[e^{\frac{-j2\pi k}{D}} z^{\frac{1}{D}}\right]^{-n}$$

SPECTRUM OF DOWNSAMPLER

On substituting, $z = e^{j\omega}$ in equation we get,

$$Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{-j2\pi k/D}\, e^{j\omega/D}\right)$$

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D}\sum_{k=0}^{D-1} X\left(e^{j(\omega-2\pi k)/D}\right)$$

On expanding above equation

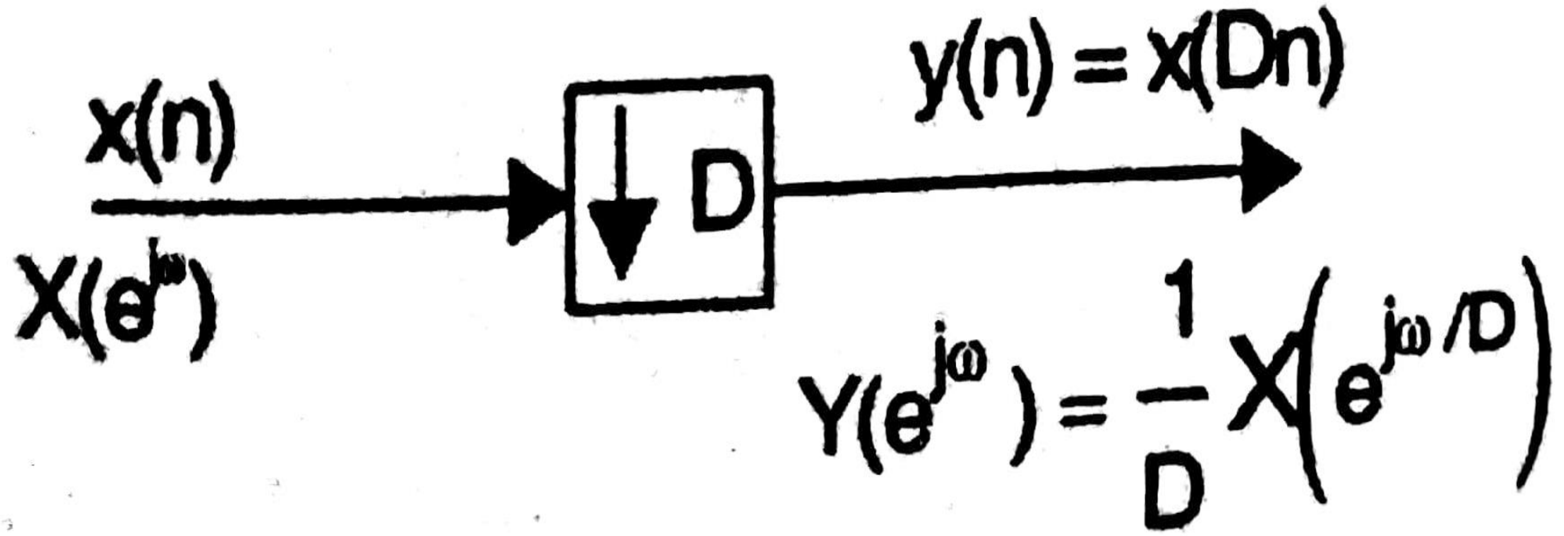
$$Y(e^{j\omega}) = \frac{1}{D} X\left(e^{j\omega/D}\right) + \frac{1}{D} X\left(e^{j(\omega-2\pi)/D}\right) + \frac{1}{D} X\left(e^{j(\omega-4\pi)/D}\right) + \ldots\ldots$$

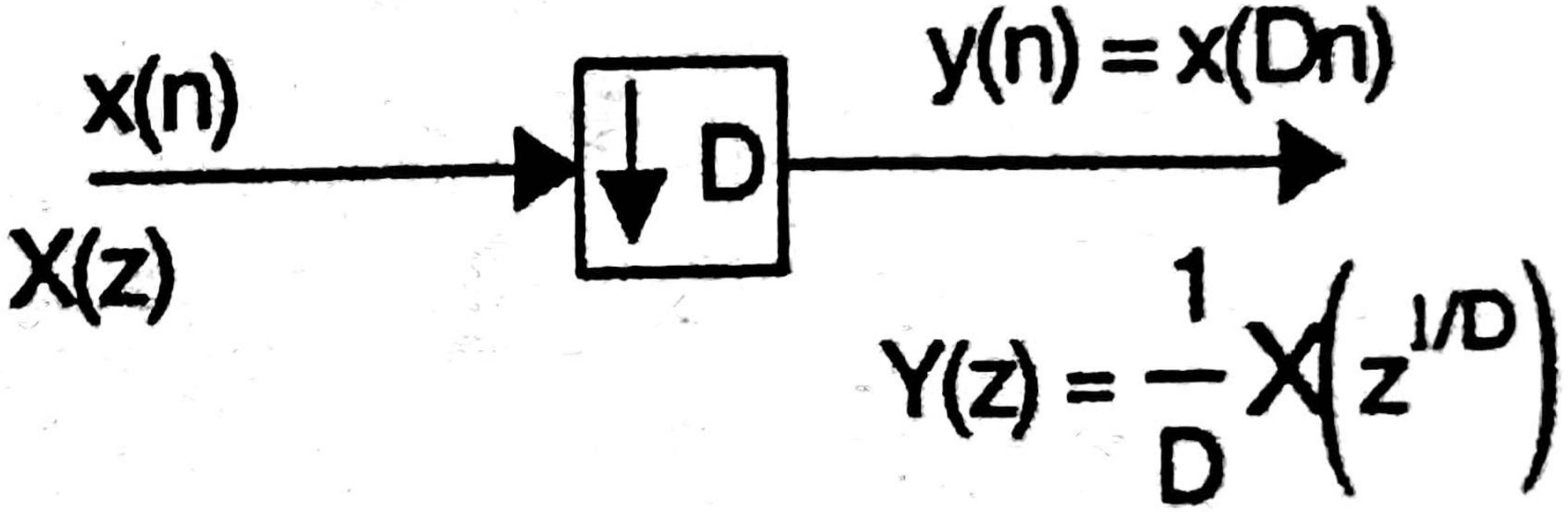
$$\ldots\ldots + \frac{1}{D} X\left(e^{j(\omega-2\pi(D-1))/D}\right)$$

SPECTRUM OF DOWNSAMPLER

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} X\left(e^{j\omega/D}\right)$$

$$Y(z) = \frac{1}{D} X\left(z^{1/D}\right)$$



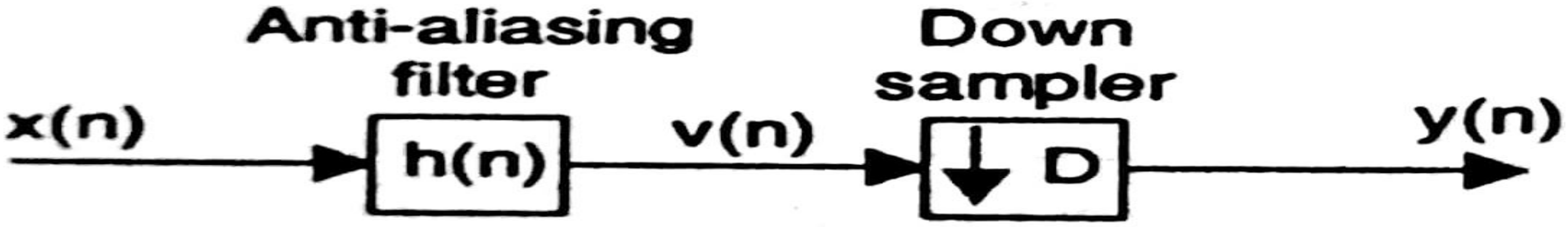


Frequency Domain Representation of downsampler

Z-Domain Representation of downsampler

ANTI-ALIASING FILTER

- When the input signal to the decimator is not bandlimited then the spectrum of decimated signal has aliasing. In order to avoid aliasing the input signal should be bandlimited to **π/D** for decimation by a factor D
- Hence the input signal is passed through a lowpass filter with a bandwidth of **π/D** before decimation. Since this lowpass filter is designed to avoid aliasing in the output spectrum of decimator, it is called anti-aliasing filter



SPECTRUM OF UPSAMPLER

- Let x(n) be an input signal to the upsampler and y(n) be the output signal
- Let x(n/I) be the upsampled version of x(n) by an integer factor I

$$\mathbf{y(n) = x(n/I)}$$

- By definition of Z-transform, y(n) can be expressed as

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} \\ &= \sum_{n=-\infty}^{+\infty} x\left(\frac{n}{I}\right) z^{-n} \\ &= \sum_{m=-\infty}^{+\infty} x(m)\, z^{-mI} \\ &= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-nI} \\ &= \sum_{n=-\infty}^{+\infty} x(n)\, (z^{I})^{-n} \end{aligned}$$

On substituting $y(n) = x\left(\frac{n}{I}\right)$ from equation

Let, $m = \frac{n}{I} \Rightarrow n = mI$

when $n = -\infty, \quad m = -\infty$

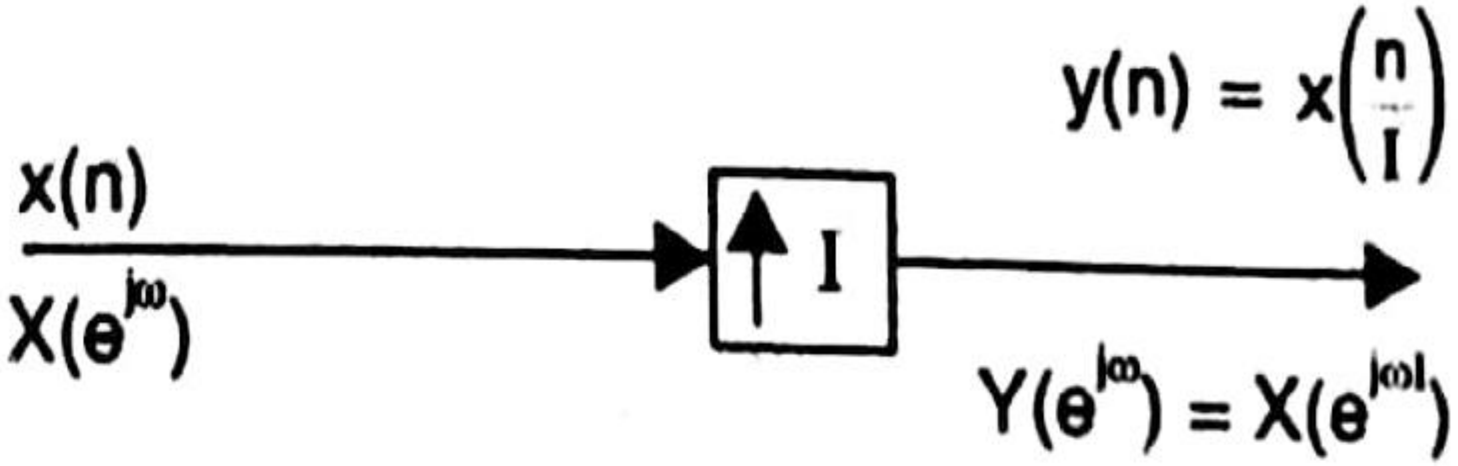
when $n = +\infty, \quad m = +\infty$

Let, $m \rightarrow n$

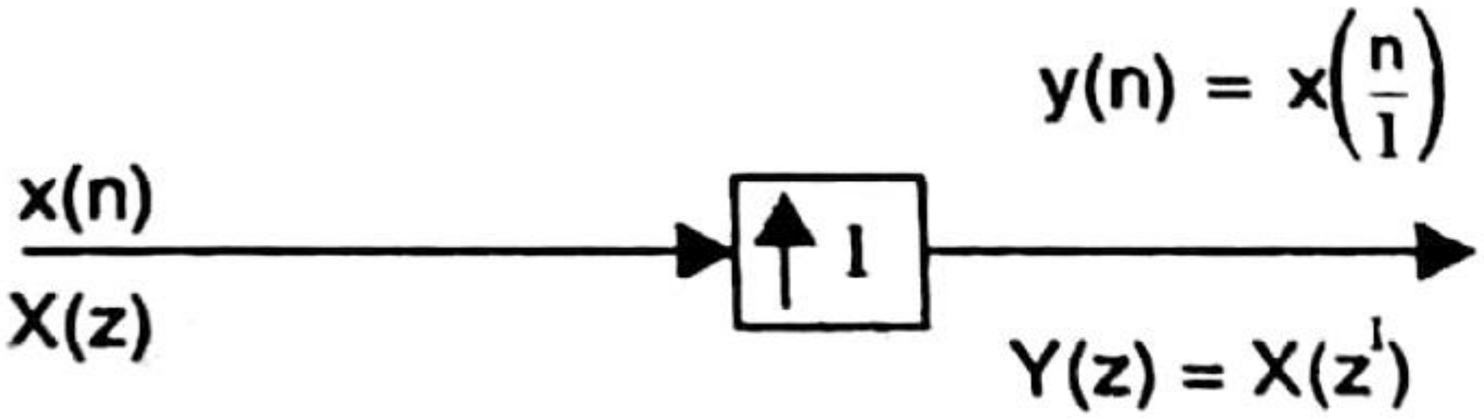
SPECTRUM OF UPSAMPLER

$$Y(e^{j\omega}) = X(e^{j\omega I})$$

$$Y(z) = X(z^{I})$$



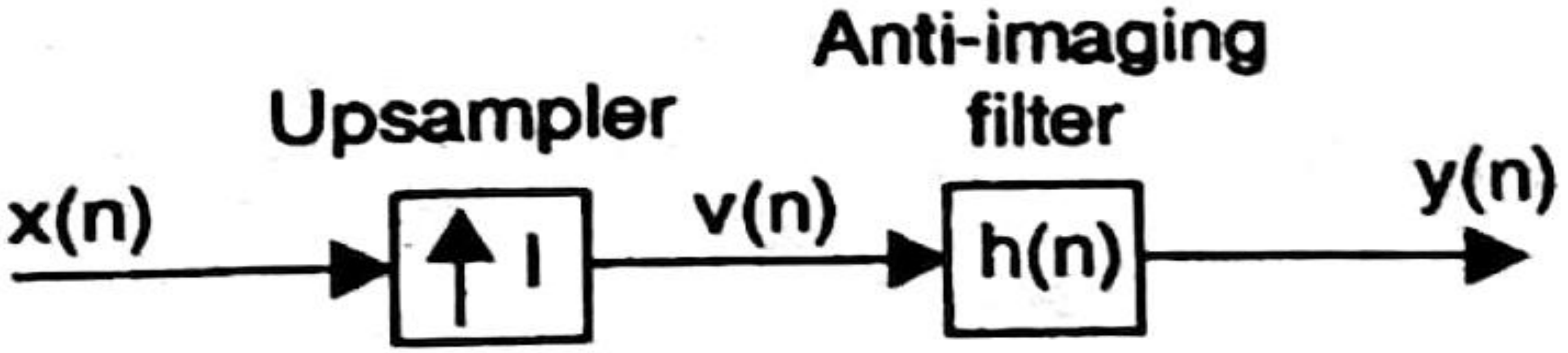
Frequency Domain Representation of upsampler



Z-Domain Representation of upsampler

ANTI-IMAGING FILTER

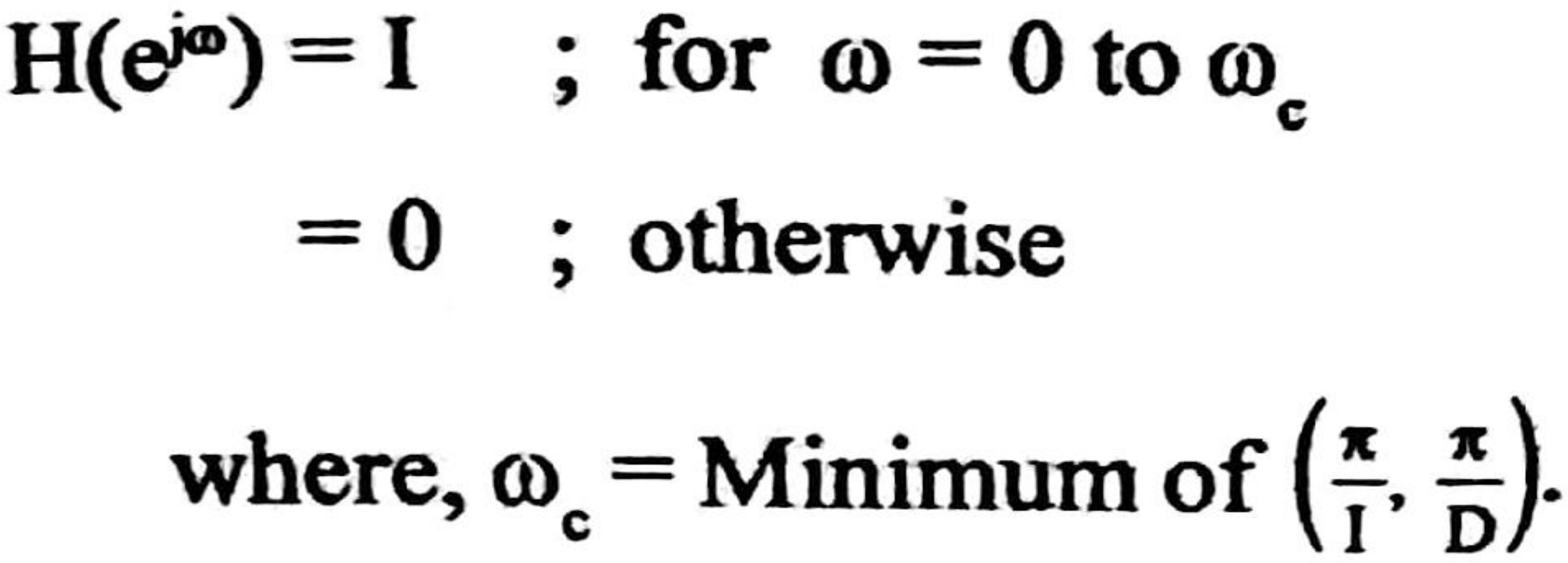
- The output spectrum of interpolator is compressed version of the input spectrum, Therefore, the spectrum of upsampled signal has multiple images in the period of 2**π**
- When upsampled by a factor of I, the output spectrum will have I images in a period of 2**π**, with each image bandlimited to **π/I**. Since the frequency spectrum in the range 0 to **π/I** are unique, we have to filter the other images
- Hence the output of upsampler is passed through a lowpass filter with a bandwidth of **π/I** . Since this lowpass filter is designed to avoid multiple images in the output spectrum, it is called anti-imaging filter



SPECTRUM OF SAMPLING RATE CONVERTER BY A RATIONAL FACTOR I/D

- h(l) = Impulse response of the lowpass filter
- $H(e^{j\omega})$ = Frequency response of the lowpass filter
- The lowpass filter is designed to have a cutoff frequency ω_c which is given

by minimum among **π/I, π/D**



$$H(e^{j\omega}) = I \quad ; \text{ for } \omega = 0 \text{ to } \omega_c$$

$$= 0 \quad ; \text{ otherwise}$$

$$\text{where, } \omega_c = \text{Minimum of } \left(\frac{\pi}{I}, \frac{\pi}{D}\right).$$

SPECTRUM OF SAMPLING RATE CONVERTER BY A RATIONAL FACTOR I/D

- Therefore, the output spectrum of lowpass filter $W(e^{j\omega})$ can be written as

$$W(e^{j\omega}) = H(e^{j\omega})\, V(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega I})$$

$$= \begin{cases} I\, X(e^{j\omega I}) & ; \quad \text{for } \omega = 0 \text{ to } \omega_c \\ 0 & ; \quad \text{otherwise} \end{cases}$$

- For decimation by an integer factor D, the frequency spectrum of output signal of decimator is given by

$$\therefore\ Y(e^{j\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} W\left(e^{j(\omega - 2\pi k/D)}\right)$$

SPECTRUM OF SAMPLING RATE CONVERTER BY A RATIONAL FACTOR I/D

- Since there is no aliasing in the output, the above equation can be evaluated for k=0 alone

$$\therefore Y(e^{j\omega}) = \frac{1}{D} W\left(e^{j\omega/D}\right)$$

$$= \begin{cases} \frac{I}{D} X(e^{j\omega I/D}) & ; \quad \text{for } \omega = 0 \text{ to } \omega_y \\ 0 & ; \quad \text{otherwise} \end{cases}$$

where, ω_y is minimum of $\left(\frac{\pi D}{I}, \pi\right)$.

- The above eqn is the spectrum of sampling rate converter by a rational factor I/D

ASSESSMENT

1. Define multirate DSP.
2. List the two ways for sampling rate conversion in the digital domain.
3. What is meant by downsampling and upsampling?
4. Low pass filter is designed to avoid multiple images in the output spectrum, it is called ----------------------
5. Define anti-aliasing Filter.
6. The lowpass filter is designed to have a cutoff frequency ω_c which is given by minimum among -------------- & ---------------

THANK YOU