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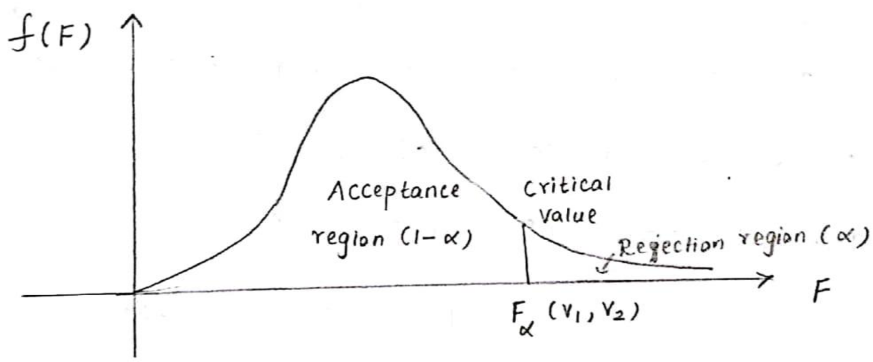
Snedecor's F-test of Significance

Snedecor's F-distribution:

A random variable F is said to follow Snedecor's F-distribution or simply F-distribution, if its probability density function is given by,

$$\boxed{f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2} \cdot F^{\frac{\nu_1}{2}-1}}{\beta\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)\left(1+\frac{\nu_1 F}{\nu_2}\right)^{(\nu_1+\nu_2)/2}}}, \quad F > 0$$

Graph of F-distribution:



Properties:

1. The square of the t-variate with n degrees of freedom follows a F-distribution with 1 and n degrees of freedom.

2. The mean of the F-distribution is $\frac{v_2}{v_2 - 2}$ $(v_2 > 2)$.

3. The variance of the F-distribution is,

$$\frac{2 v_2^2 (v_1 + v_2 - 2)}{v_1 (v_2 - 2)^2 (v_2 - 4)} \quad (v_2 > 4).$$

Applications (uses) of F-distribution:

F-test is used to test

(i) whether two independent samples have been drawn from the normal populations with the same variance σ^2

(ii) whether the two independent estimates of the population variance are homogeneous or not.

F-test of Significance of the difference between Population variances and F-table.

If s_1^2 and s_2^2 are the variances of two samples of sizes n_1 and n_2 respectively, the estimates of the population variance based on these samples are respectively,

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \quad \text{and} \quad S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

The quantities $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ are called the degrees of freedom of these estimates.
We want to test if these estimates S_1^2 and S_2^2 are significantly different or if the samples may be regarded as drawn from the same population or from two populations with same variance σ^2.

Let $$\boxed{F = \frac{S_1^2}{S_2^2} = \frac{\dfrac{n_1 s_1^2}{n_1 - 1}}{\dfrac{n_2 s_2^2}{n_2 - 1}}}$$ and $S_1^2 > S_2^2$

If $S_1^2 = S_2^2$, then $F = 1$. Hence, our object is to find how far any observed value of F differs from unity, consistent with our assumption of the equality of the population variances.

Note :

1. $F > 0$ always.
2. Suppose $S_2^2 > S_1^2$, then $F = \dfrac{S_2^2}{S_1^2}$ with $v_1 = n_2 - 1$ and $v_2 = n_1 - 1$ degrees of freedom.
   and $S_1^2 = \dfrac{\Sigma (x - \bar{x})^2}{n_1 - 1}$, $S_2^2 = \dfrac{\Sigma (x - \bar{x})^2}{n_2 - 1}$

Problems :

(1) In a sample of 8 observations, the sum of the squares of deviations of the sample values from the sample mean was 94.5 and in another sample of 10 observations, it was found to be 101.7. Test whether the difference of variances is significant.

Solution :

Given : $n_1 = 8$, $n_2 = 10$

$$\Sigma (x_1 - \bar{x}_1)^2 = 94.5 \; , \; \Sigma (x_2 - \bar{x}_2)^2 = 101.7$$

Null hypothesis : H_0 : The difference of variances is not significant. i.e., $H_0 : \sigma_1^2 = \sigma_2^2$

Alternative hypothesis : $H_1 : \sigma_1^2 \neq \sigma_2^2$ (Two - tailed test)

Calculation of population variances :

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{94.5}{7} = 13.5$$

$$S_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1} = \frac{101.7}{9} = 11.3$$

Test Statistic : Since $S_1^2 > S_2^2$,

$$F = \frac{S_1^2}{S_2^2} = \frac{13.5}{11.3}$$

$$\boxed{F = 1.195}$$

Table value :

At $\alpha = 5\%$, for $v_1 = n_1 - 1 = 7$, $v_2 = n_2 - 1 = 9$ d.o.f , the table value of F is given by,

$$\boxed{F_\alpha = 3.29}$$

Decision :

Since $F < F_\alpha$, H_0 is accepted. $\therefore$ The difference of variance is not significant at 5% LOS.

(2) Two random samples gave the following results.

Sample	Size	Sample Mean	Sum of Squares of deviations from the mean
1	12	14	108
2	10	15	90

Test whether the samples came from the same Population.

Solution :

Given : $n_1 = 12$, $n_2 = 10$

$\bar{x}_1 = 14$, $\bar{x}_2 = 15$

$$\Sigma (x_1 - \bar{x}_1)^2 = 108$$

$$\Sigma (x_2 - \bar{x}_2)^2 = 90$$

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(i) To test $\sigma_1^2 = \sigma_2^2$ (F-test)

Null hypothesis : H_0 : The two samples have been drawn from the population with equal variances i.e., $H_0 : \sigma_1^2 = \sigma_2^2$.

Alternative hypothesis : $H_1 : \sigma_1^2 \neq \sigma_2^2$ (Two-tailed test)

Calculation of population variances :

$$S_1^2 = \frac{\Sigma (x_1 - \bar{x}_1)^2}{n_1 - 1} = \frac{108}{11} = 9.82$$

$$S_2^2 = \frac{\Sigma (x_2 - \bar{x}_2)^2}{n_2 - 1} = \frac{90}{9} = 10.$$

Test-Statistic :

Since $S_2^2 > S_1^2$,

$$F = \frac{S_2^2}{S_1^2} = \frac{10}{9.82}$$

$$\boxed{F = 1.02}$$

Table value :

At $\alpha = 5\%$ LOS, $(v_1, v_2) = (n_2 - 1, n_1 - 1)$

$= (9, 11)$ d.o.f the table value of F is given by,

$$\boxed{F_\alpha = 2.90}$$

Decision :

Since $F < F_\alpha$, H_0 is accepted . Hence , samples came from the populations of equal variance .

(ii) To test $\mu_1 = \mu_2$ (t-test):

Null hypothesis : H_0 : The two samples have been drawn from the population with equal mean i.e.,

$H_0 : \mu_1 = \mu_2$.

Alternative hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two-tailed test)

Calculation of Population Variance :

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$$

$$= \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2} \qquad \left[\because S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} \text{ and } S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} \right]$$

$$= \frac{108 + 90}{20} = 9.9$$

$$S = \sqrt{9.9} = 3.15$$

$$\boxed{S = 3.15}$$

Test Statistic :

$$t = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$t = \frac{14 - 15}{(3.15)\sqrt{\frac{1}{12} + \frac{1}{10}}}$$

$$\boxed{|t| = 0.7423}$$

Table value :

At $\alpha = 5\%$ LOS , $\nu = n_1 + n_2 - 2 = 12 + 10 - 2 = 20$ d.o.f , the table value of t is given by,

$$\boxed{t_\alpha = 2.086}$$

Decision :

Since $|t| < t_\alpha$, H_0 is accepted . ∴ The Samples have been drawn from the population with equal means .

Conclusion :

Therefore , we conclude that the two samples have been drawn from the same normal population .

(3) Two random samples of 11 and 9 items show the sample standard deviations of their weights as 0.8 and 0.5 respectively . Assuming that the weight distributions are normal , test the hypothesis that the true variances are equal , against the alternative hypothesis that they are not .

Solution :

Given: $n_1 = 11$, $n_2 = 9$

$s_1 = 0.8$, $s_2 = 0.5$

Null hypothesis : H_0 : The variances are equal. i.e.,

$$H_0 : \sigma_1^2 = \sigma_2^2$$

Alternative hypothesis : $H_1 : \sigma_1^2 \neq \sigma_2^2$

~~Test~~ Calculation of Population Variances :

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{11 \times 0.8^2}{11-1} = \frac{7.04}{10}$$

$$\boxed{S_1^2 = 0.704}$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{9 \times 0.5^2}{9-1} = \frac{2.25}{8}$$

$$\boxed{S_2^2 = 0.2813}$$

Test- Statistic :

$$F = \frac{S_1^2}{S_2^2} = \frac{0.704}{0.2813} \quad (\because S_1^2 > S_2^2)$$

$$\boxed{F = 2.503}$$

Table value :

At $\alpha = 5\%$ LOS , $v_1 = n_1 - 1 = 11 - 1 = 10$, $v_2 = n_2 - 1 = 9 - 1 = 8$ d.o.f , the table value of

F is given by,

$$\boxed{F_{\alpha} = 3.34}$$

Decision:

Since $F < F_{\alpha}$, H_0 is accepted. ∴ The true variances are equal.