Type - 2 Poisson Equation.

WKT $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

(or)

$\nabla^2 u = f(x, y)$

Then the Standard five point formula is.

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}$

$= h^2 f(ih, jh)$

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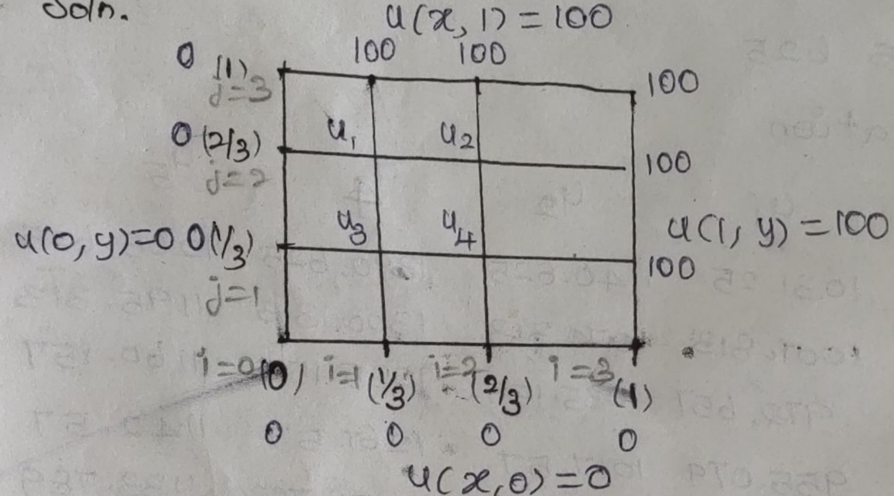
1. Solve the Poisson eqn. $\nabla^2 u = -81xy$, $0 < x < 1$, $0 < y < 1$ and $u(0, y) = u(x, 0) = 0$; $u(x, 1) = u(1, y) = 100$ with the square with each of length $h = 1/3$

Soln.



Given $f(x, y) = -81xy$ and $h = 1/3$

By standard five point formula,

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}$

$= h^2 [-81 (ih)(jh)]$

$= \frac{1}{9} [-81 ij\, h^2] = \frac{1}{81} (-81 ij)$

$= -ij \longrightarrow (1)$

For u_1 $(i=1, j=2)$,

$(1) \Rightarrow u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = (1)(2)$

$0 + u_2 + u_3 + 100 - 4u_1 = -2$

$-4u_1 + u_2 + u_3 = -102 \rightarrow (2)$

For u_2 $(i=2, j=2)$,

$(1) \Rightarrow u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -2(2)$

$u_1 + 100 + u_4 + 100 - 4u_2 = -4$

$u_1 + u_4 - 4u_2 = -204 \rightarrow (3)$

For u_3 $(i=1, j=1)$,

$(1) \Rightarrow u_{0,1} + u_{2,1} + u_{1,2} + u_{1,0} - 4u_{1,1} = -(1)(1)$

$0 + u_4 + u_1 + 0 - 4u_3 = -1$

$u_1 + u_4 - 4u_3 = -1 \rightarrow (4)$

For u_4 $(i=2, j=1)$

$(1) \Rightarrow u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -(2)(1)$

$u_3 + 100 + 0 + u_2 - 4u_4 = -2$

$u_2 + u_3 - 4u_4 = -102 \rightarrow (5)$

Solve (2) & (5),

$-4u_1 + u_2 + u_3 + 0u_4 = -102 \rightarrow (2)$

$0u_1 + u_2 + u_3 - 4u_4 = -102 \rightarrow (5)$

$(-) \quad (-) \quad (-) \quad (+) \quad (+)$

$(2)-(5) \Rightarrow -4u_1 + 4u_4 = 0$

$\Rightarrow -4u_1 = -4u_4$

$\Rightarrow u_1 = u_4 \rightarrow (6)$

Solve (3) & (4),

$u_1 - 4u_2 + 0u_3 + u_4 = -204 \rightarrow (3)$

$u_1 + 0u_2 - 4u_3 + u_4 = -1 \rightarrow (4)$

$(-) \quad (-) \quad (+) \quad (-) \quad (+)$

$(3)-(4) \quad -4u_2 + 4u_3 = -203 \rightarrow (7)$

Subs. $u_1 = u_4$ in (3)

$$u_1 + u_1 - 4u_2 = -204$$

$$2u_1 - 4u_2 = -204 \rightarrow (8)$$

$$(8)\times 2 \Rightarrow 4u_1 - 8u_2 = -408$$

$$(2) \Rightarrow -4u_1 + u_2 + u_3 = -102$$

$$-7u_2 + u_3 = -510 \rightarrow (9)$$

Solve (7) & (9)

$$(7) \Rightarrow -4u_2 + 4u_3 = -203$$

$$(9)\times 4 \Rightarrow \underset{(+)}{-}28u_2 \underset{(-)}{+} 4u_3 = \underset{(+)}{-}2040$$

$$24u_2 = 1837$$

$$u_2 = 76.542$$

Subs u_2 in (7),

$$-4(76.542) + 4u_3 = -203$$

$$4u_3 = -203 + 306.168$$

$$= 25.792$$

$$u_3 = 25.792$$

Subs. u_2 in (8),

$$2u_1 - 4u_2 = -204$$

$$2u_1 - 4(76.542) = -204$$

$$2u_1 = -204 + 4(76.542)$$

$$= 102.168$$

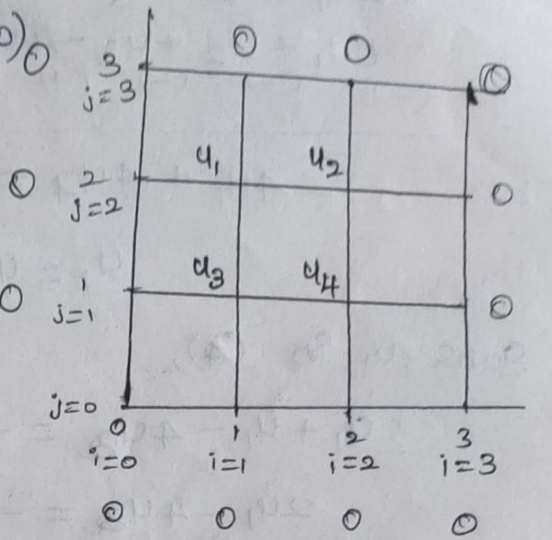
$$u_1 = 51.084$$

2. Solve for the poisson eqn. $\nabla^2 u = -10(x^2+y^2+10)$ over $x=0, y=0$ & $x=3$ and $y=3$ with $u=0$ on the boundary, taking $h=1$

Solution :

Given $F(x,y) = -10(x^2+y^2+10)$

and $h=1$



By standard five point formula,

$u_{i-1,j} + u_{i+1,j} + u_{i,j-1}$

$+ u_{i,j+1} - 4u_{i,j}$

$= h^2[-10((ih)^2 + (jh)^2 + 10)]$

$= 1[-10(i^2 + j^2 + 10)]$

$= -10(i^2+j^2+10) \rightarrow (1)$

For u_1 $(i=1, j=2)$,

$(1) \Rightarrow u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = -10(1+4+10)$

$0 + u_2 + u_3 + 0 - 4u_1 = -150$

$u_2 + u_3 - 4u_1 = -150 \rightarrow (2)$

For u_2 $(i=2, j=2)$,

$(1) \Rightarrow u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = -10(4+4+10)$

$u_1 + 0 + u_4 + 0 - 4u_2 = -180$

$u_1 + u_4 - 4u_2 = -180 \rightarrow (3)$

For u_3 $(i=1, j=1)$

$(1) \Rightarrow u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = -10(1+1+10)$

$0 + u_4 + 0 + u_1 - 4u_3 = -120$

$u_1 + u_4 - 4u_3 = -120 \rightarrow (4)$

For u_4 $(i=2, j=1)$

$(1) \Rightarrow u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} = -10(4+1+10)$

$u_3 + 0 + 0 + u_2 - 4u_4 = -150$

$u_2 + u_3 - 4u_4 = -150 \rightarrow (5)$

Solve (2) & (5),

$$-4u_1 + u_2 + u_3 + 0u_4 = -150$$
$$0u_1 + u_2 + u_3 - 4u_4 = -150$$
$$(-) \quad (-) \quad (-) \quad (+) \quad (+)$$

$$-4u_1 + 4u_4 = 0$$
$$u_1 = u_4 \rightarrow (6)$$

Subs u_1 in (3),

$$u_1 + u_1 - 4u_2 = -180$$
$$2u_1 - 4u_2 = -180 \rightarrow (7)$$

Solve (2) & (7)

(7)×2 ⇒ $4u_1 - 8u_2 = -360$

(2) ⇒ $-4u_1 + u_2 + u_3 = -150$

$$-7u_2 + u_3 = -510 \rightarrow (8)$$

Solve (3) & (4),

$$u_1 - 4u_2 + 0u_3 + u_4 = -180$$
$$u_1 + 0u_2 - 4u_3 + u_4 = -120$$
$$(-) \quad (-) \quad (+) \quad (-) \quad (+)$$

$$-4u_2 + 4u_3 = -60 \rightarrow (9)$$

Solve (8) & (9)

$$-7u_2 + u_3 = -510 \rightarrow (8)$$
$$-4u_2 + 4u_3 = -60 \rightarrow (9)$$

(8)×4 ⇒ $-28u_2 + 4u_3 = -2040$

$$-4u_2 + 4u_3 = -60$$
$$(+) \quad (-) \quad (+)$$

$$-24u_2 = -1980$$
$$u_2 = 1980/24$$
$$u_2 = 82.5$$

Subs u_2 in (9),

$$-4(82.5) + 4u_3 = -60$$
$$4u_3 = -60 + 4(82.5)$$
$$u_3 = 270/4 = 67.5$$

Subs. u_2 & u_3 in (2),

$$-4u_1 + u_2 + u_3 = -150$$

$$4u_1 = u_2 + u_3 + 150$$

$$= 82.5 + 67.5 + 150$$

$$4u_1 = 300$$

$$u_1 = 75$$

$\therefore u_1 = u_4 = 75$

Hence $u_1 = 75$

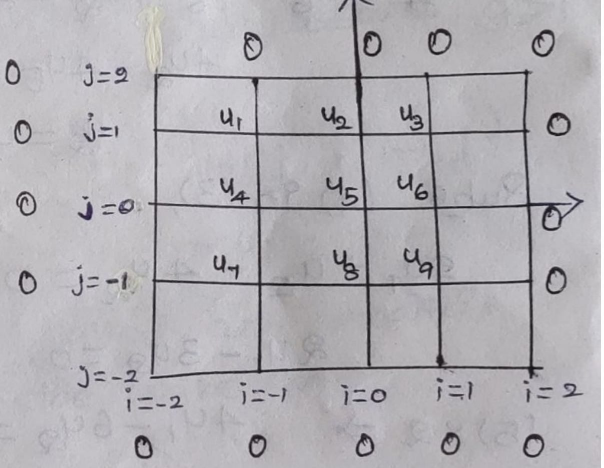
$u_2 = 82.5$

$u_3 = 67.5$

$u_4 = 75$

5]. Solve the poisson eqn. $\nabla^2 u = 8x^2y^2$ inside a square region the line $x = \pm 2, y = \pm 2$ and $u = 0$ on boundary. Assume centre of the square and divide square into 16 equal parts.

Soln.



Assume $h = 1$

Given $F(x,y) = 8x^2y^2$

By Standard five point formula,

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}$$

$$= h^2[8(ih)^2(jh)^2]$$

$$= 8i^2h^2j^2h^2$$

$$= 8i^2j^2 \quad \because h = 1$$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8i^2j^2 \quad \to (1)$$

Given problem is symmetric w.r.to x & y axis.

$\therefore u_1 = u_3 = u_7 = u_9$ and $u_2 = u_4 = u_6 = u_8$.

To find u_1, u_2, u_5 only

For u_1 $(i=-1, j=1)$,

$u_{-2,1} + u_{0,1} + u_{-1,0} + u_{-1,2} - 4u_{-1,1} = 8(-1)^2(1)$

$0 + u_2 + u_4 + 0 - 4u_1 = 8$

$2u_2 - 4u_1 = 8 \rightarrow (2) \quad (\because u_2 = u_4)$

For u_2 $(i=0, j=1)$ in (1),

$u_{-1,1} + u_{1,1} + u_{0,0} + u_{0,2} - 4u_{0,1} = 8(0)$

$u_1 + u_3 + u_5 + 0 - 4u_2 = 0$

$2u_1 + u_5 - 4u_2 = 0 \rightarrow (3) \quad (\because u_1 = u_3)$

For u_5 $(i=0, j=0)$ in (1),

$u_{-1,0} + u_{1,0} + u_{0,-1} + u_{0,1} - 4u_{0,0} = 8(0)$

$u_4 + u_6 + u_8 + u_2 - 4u_5 = 0$

$4u_2 - 4u_5 = 0 \quad (\because u_2 = u_4 = u_6 = u_8)$

$u_2 = u_5 \rightarrow (4)$

Sub. (4) in (3),

$2u_1 + u_2 - 4u_2 = 0$

$2u_1 - 3u_2 = 0 \rightarrow (5)$

$(5)\times 2 \Rightarrow \quad 4u_1 - 6u_2 = 0$

$(2) \quad \Rightarrow \quad -4u_1 + 2u_2 = 8$

$-4u_2 = 8$

$u_2 = -2$

$(4) \Rightarrow \quad u_2 = u_5 = -2$

$(5) \Rightarrow \quad 2u_1 - 3u_2 = 0$

$2u_1 - 3(-2) = 0 \quad \Rightarrow 2u_1 = -6$

$u_1 = -3$

$\therefore \quad u_1 = u_3 = u_7 = u_9 = -3$

and $\quad u_2 = u_4 = u_6 = u_8 = -2$

$$u_5 = -2$$