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Euler's method :

$$y_{n+1} = y_n + h\,f(x_n, y_n) \;,\; n = 0, 1, 2, \dots$$

Modified Euler's method :

$$y_{n+1} = y_n + \frac{1}{2} h \left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

Improved Euler's method :

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + h\,f(x_n, y_n))\right]$$

Problems :

(1) Using Euler's method find the solution of the initial value problem $\frac{dy}{dx} = \log(x+y)$, $y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.

Solution :

Given : $f(x,y) = \log(x+y)$

$x_0 = 0$, $y_0 = 2$, $x_1 = 0.2$, $h = 0.2$

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

For $n = 0$, $y_1 = y_0 + h\,f(x_0, y_0)$

$$= 2 + (0.2)\log(x_0 + y_0)$$

$$= 2 + (0.2)\log(0 + 2)$$

$$= 2 + (0.2)\log 2$$

$$= 2 + (0.2)(0.3010)$$

$$y_1 = 2.0602$$

i.e., $\boxed{y(0.2) = 2.0602}$

(2) Using Euler's method find $y(0.2)$ and $y(0.4)$ from $\frac{dy}{dx} = x+y$, $y(0) = 1$ with $h = 0.2$

Solution:

Given: $f(x,y) = x+y$

$x_0 = 0$, $y_0 = 1$, $h = 0.2$

$x_1 = 0.2$, $y_1 = ?$

$x_2 = 0.4$, $y_2 = ?$

By Euler's formula,

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 1 + (0.2)(x_0 + y_0)$$

$$= 1 + (0.2)(0+1)$$

$$\boxed{y(0.2) = 1.2}$$

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$= 1.2 + (0.2)(x_1 + y_1)$$

$$= 1.2 + (0.2)(0.2 + 1.2)$$

$$= 1.2 + 0.28$$

$$\boxed{y(0.4) = 1.48}$$

(3) Compute y at $x = 0.25$ by modified Euler method given $y' = 2xy$, $y(0) = 1$

Solution:

Given: $x_0 = 0$, $y_0 = 1$

$x_1 = 0.25$, $y_1 = ?$

$h = 0.25$

(6)

$f(x, y) = 2xy$

By modified Euler's method,

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right]$$

$$y_1 = y_0 + h f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

$$= 1 + (0.25) f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2} f(0,1)\right)$$

$$= 1 + (0.25) f\left(0.125, 1 + (0.125)(0)\right)$$

$$= 1 + (0.25) f(0.125, 1)$$

$$= 1 + (0.25)(2 \times 0.125 \times 1)$$

$$\boxed{y(0.25) = 1.0625}$$

(4) Using modified Euler's method, compute $y(0.1)$ with $h = 0.1$ from $y' = y - \frac{2x}{y}$, $y(0) = 1$

Solution:

Given : $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$x_1 = 0.1$

$$f(x, y) = y - \frac{2x}{y}$$

By modified Euler's method,

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

$$y_1 = y_0 + h f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

$$= 1 + (0.1) f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} f(0,1)\right)$$

$$= 1 + (0.1) f\left(0.05, 1.05\right)$$

$$= 1 + 0.1(0.9548)$$

$$\boxed{y(0.1) = 1.09548}$$