INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Suppose the first order differential equation

$$\frac{dy}{dx} = f(x,y) \rightarrow (1)$$

is given with the initial condition $y(x_0) = y_0$.

If we can obtain a formula for the solution, we may evaluate it numerically, either directly or by the use of tables. If that formula is too complicated or if no formula for the solution is available, we may apply Step-by-Step method. In this method we start from $y(x_0) = y_0$ and proceed with the approximate value of y_1 for the solution of y in (1) at $x = x_1 = x_0 + h$. In the second step we compute an approximate value y_2 of the solution at $x = x_2 = x_0 + 2h$ etc. Here h is a fixed number called stepsize.

Single Step methods (or) Pointwise methods :

A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution. The methods of Taylor and picard belong to these type.

Multistep methods (or) Step by Step methods :

In a set of tabulated values of x and y, we obtain y by iterative process. The methods of Euler, Runge-Kutta, Milne, Adam-Bashforth etc., belong to this type. Here the values of y are computed by short steps for equal intervals h of the independent variable. These values are iterated till we get the desired accuracy. Hence these methods are called step by step methods.

Taylor Series method :

$$y_1 \overset{y(x_1)=}{=} y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

where $\overset{x_1 = x_0+h}{h = x - x_0}$ and $y_1 = y(x) = y(x_0+h)$

$$y_2 \overset{y(x_2)=}{=} y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

where $\overset{x_2 = x_1+h}{h = \frac{x - x_0}{2}}$ and $y_2 = y(x) = y(x_0+2h)$

$$y_3 = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \cdots$$

where $h = \frac{x - x_0}{3}$ and $y_3 = y(x) = y(x_0+3h)$

In general,

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots$$

$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots$ where $x_1 = x_0 + h$

$y_2 = y(x_2) = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \cdots$ where $x_2 = x_1 + h$

$y_3 = y(x_3) = y_2 + \frac{h}{1!} y_2' + \frac{h^2}{2!} y_2'' + \cdots$ where $x_3 = x_2 + h$

In general,

$y_{n+1} = y(x_{n+1}) = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \cdots$ where $x_{n+1} = x_n + h$

Type I: First order ODE / Given $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ (2)

(1) Using Taylor's series method, find $y(1.1)$ given $y' = x + y$, $y(1) = 0$.

Solution:

Here $x_0 = 1$, $y_0 = 0$, $h = 0.1$

$x_1 = 1.1$, $y_1 = ?$

$y' = x + y \quad ; \quad y_0' = x_0 + y_0 = 1$

$y'' = 1 + y' \quad ; \quad y_0'' = 1 + y_0' = 1 + 1 = 2$

$y''' = y'' \quad ; \quad y_0''' = y_0'' = 2$

$y^{iv} = y''' \quad ; \quad y_0^{iv} = y_0''' = 2$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$= 0 + 0.1(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24} \times 2$$

$= 0.1 + 0.01 + 0.0003 + 0.0000083$ terms

$\boxed{y(1.1) = 0.1103}$

(2) By means of Taylor series expansion, find y at $x = 0.1, 0.2$ correct to three significant digits given $\frac{dy}{dx} - 2y = 3e^x$, $y(0) = 0$.

Solution:

Here $x_0 = 0$, $y_0 = 0$

$x_1 = 0.1$, $x_2 = 0.2$, $y_1 = ?$, $y_2 = ?$

$y' - 2y = 3e^x$

i.e., $y' = 3e^x + 2y$

$$y' = 3e^{x} + 2y \quad ; \quad y_0' = 3 + 2(0) = 3$$
$$y'' = 3e^{x} + 2y' \quad ; \quad y_0'' = 3 + 2(3) = 9$$
$$y''' = 3e^{x} + 2y'' \quad ; \quad y_0''' = 3 + 2(9) = 21$$
$$y^{IV} = 3e^{x} + 2y''' \quad ; \quad y_0^{IV} = 3 + 2(21) = 45$$

$$\therefore \quad y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$= 0 + 0.1\,(3) + \frac{(0.1)^2}{2}(9) + \frac{(0.1)^3}{6}(21) + \frac{(0.1)^4}{24} 45 + \cdots$$

$$= 0.3 + 0.045 + 0.0035 + 0.0001875 + \cdots$$

$$\boxed{y(0.1) = 0.349}$$

$$x_1 = 0.1 \;, \quad y_1 = 0.349$$

$$y_1' = 3e^{x_1} + 2y_1 = 4.0135$$
$$y_1'' = 3e^{x_1} + 2y_1' = \cancel{11.0258}\ 11.3425$$
$$y_1''' = 3e^{x_1} + 2y_1'' = \cancel{25.3671}\ 26.0005$$
$$y_1^{IV} = 3e^{x_1} + 2y_1''' = \cancel{54.0497}\ 55.3165$$

$$\therefore \quad y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \cdots$$

$$= 0.349 + (0.1)(4.0135) + \frac{(0.1)^2}{2}(11.\overset{3425}{\cancel{0258}})$$
$$+ \frac{(0.1)^3}{6}(\overset{26.0005}{\cancel{25.3671}}) + \frac{(0.1)^4}{24}(\overset{55.3165}{\cancel{54.0497}}) + \cdots$$

$$= 0.349 + 0.40135 + \overset{0.05671}{\cancel{0.05513}} + \overset{0.00000641}{\cancel{0.0042279}}$$
$$+ \cancel{0.000225}$$

$$= 0.80099$$

$$\boxed{y(0.2) = 0.8071}$$

Given: $\frac{dy}{dx} = f(x,y,z)$ ③
$\frac{dz}{dx} = g(x,y,z)$
$y(x_0) = y_0$
$z(x_0) = z_0$

Type 2: First order simultaneous ODE

① Solve the system of equations $\frac{dy}{dx} = z - x^2$, $\frac{dz}{dx} = y + x$ with $y(0) = 1$, $z(0) = 1$ by taking $h = 0.1$, to get $y(0.1)$ and $z(0.1)$. Here y and z are dependent variables and x is independent.

2, 4, 27, 32, 35, 36, 118

Solution:

Here $x_0 = 0$, $y_0 = 1$, $z_0 = 1$, $h = 0.1$

$y' = z - x^2$	$y_0' = z_0 - x_0^2 = 1 - 0 = 1$	$z' = y + x$	$z_0' = y_0 + x_0 = 1 + 0 = 1$
$y'' = z' - 2x$	$y_0'' = z_0' - 2x_0 = 1 - 0 = 1$	$z'' = y' + 1$	$z_0'' = y_0' + 1 = 1 + 1 = 2$
$y''' = z'' - 2$	$y_0''' = z_0'' - 2 = 2 - 2 = 0$	$z''' = y''$	$z_0''' = y_0'' = 1$
$y^{iv} = z'''$	$y_0^{iv} = z_0''' = 1$	$z^{iv} = y'''$	$z_0^{iv} = y_0''' = 0$

By Taylor's series,

$$y_1 = y(0.1) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1)$$

$$= 1 + 0.1 + 0.005 + 0.0000042$$

$$\boxed{y(0.1) = 1.105}$$

$$z_1 = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \dots$$

$$= 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{6}(1) + 0$$

$$\doteq 1 + 0.1 + 0.01 + 0.000167$$

$$\boxed{z(0.1) = 1.1102}$$

(2) Using Taylor series method, find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$

$x_0 = 0$, $y_0 = 2$, $z_0 = 1$

$y' = x + z \qquad z' = x - y^2$

$y'' = 1 + z' \qquad z'' = 1 - 2yy'$

$y''' = z'' \qquad z''' = -2(yy'' + y'^2)$

$y_0' = 1 \qquad z_0' = -4$

$y_0'' = -3 \qquad z_0'' = -3$

$y_0''' = -3 \qquad z_0''' = 10$

$\boxed{y_1 = 2.0845} \qquad \boxed{z_1 = 0.5867}$

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$$y = \frac{4 \times 1 \times -1}{-1 \times -4 \times -6}(12) + \frac{5 \times 1 \times -1}{1 \times -3 \times -5}(13) +$$

$$\frac{5 \times 4 \times -1}{4 \times 3 \times -2}(14) + \frac{5 \times 4 \times 1}{6 \times 5 \times 2}(16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{44}{3}$$

$$\boxed{y = 14.6667}$$

(4) Find the missing term in the following table using Lagrange's interpolation.

x	0	1	2	3	4
y	1	3	9	—	81

Solution:

By lagrange's interpolation formula,

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$

$y_0 = 1$, $y_1 = 3$, $y_2 = 9$, $y_3 = 81$

& $x = 3$

Type 3: Second order ODE   Given : $y'' = f(x, y, y')$, $y(x_0) = y_0$, $y'(x_0) = y_0'$ (4)

(1) By Taylor's series method find $y(0.1)$ given that $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$.

Solution:

Here $x_0 = 0$, $y_0 = 1$, $y_0' = 0$

$y'' = y + xy'$; $y_0'' = y_0 + x_0 y_0' = 1 + 0 = 1$

$y''' = y' + xy'' + y'$; $y_0''' = 2y_0' + x_0 y_0'' = 2(0) + 0 = 0$

$= 2y' + xy''$

$y^{iv} = 2y'' + xy''' + y''$; $y_0^{iv} = 2y_0'' + x_0 y_0''' + y_0''$
$= 2(1) + 0 + 1 = 3$

$$\therefore y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \cdots$$

$$= 1 + \frac{(x - 0)}{1}(0) + \frac{(x - 0)^2}{2}(1) + 0 + \frac{(x - 0)^4}{4!}(3) + \cdots$$

$$y(x) = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

$$y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + \cdots$$

$= 1 + 0.005 + 0.0000125$

$= 1.0050125$

$$\boxed{y(0.1) = 1.0050}$$

2, L16, L18

(2) Evaluate the values of $y(0.1)$ and $y(0.2)$ given $y'' - x(y')^2 + y^2 = 0$; $y(0) = 1$, $y'(0) = 0$ by using Taylor series method.

<u>Solution:</u>

Here $x_0 = 0$, $y_0 = 1$, $y_0' = 0$.

$y'' = x(y')^2 - y^2$; $y_0'' = x_0(y_0')^2 - y_0^2 = -1$

$y''' = x \cdot 2y'y'' + (y')^2 - 2yy'$; $y_0''' = 0$

$$y^{iv} = 2xy'y''' + 2xy''y'' + 2y'y'' + 2y'y'' - 2yy'' - 2y'y'$$

$$= 2xy'y''' + 4y'y'' + 2x(y'')^2 - 2yy'' - 2(y')^2 \quad ; \quad y_0^{iv} = 2$$

$$\therefore y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \cdots$$

$$= 1 + \frac{x}{1!}(0) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(0) + \frac{x^4}{4!}(2) + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{(0.1)^4}{12} + \cdots$$

$$= 1 - 0.005 + 0.000008 = 0.995008$$

$$\boxed{y(0.1) = 0.995}$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{12} + \cdots$$

$$= 1 - 0.02 + 0.00013 = 0.980133$$

$$\boxed{y(0.2) = 0.9801}$$