Fouth order Runge_Kutta method for Solving First and Second order Equations:

Second order R_K method

$$K_1 = hf(x, y)$$

$$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$$

$\Delta y = K_2$ where $h = \Delta x$; $y_1 = y_0 + \Delta y$

Third order R_K method:

$$K_1 = hf(x, y)$$

$$K_2 = hf\left(x+\frac{h}{2}, y+\frac{K_1}{2}\right)$$

$$K_3 = hf[x+h, y+2K_2-K_1]$$

$$\Delta y = \frac{1}{6}[K_1 + 4K_2 + K_3]$$

Now $y_1 = y_0 + \Delta y$

Fourth order R_K method:

$$K_1 = hF(x, y)$$

$$K_2 = hf\left[x+\frac{h}{2}, y+\frac{K_1}{2}\right]$$

$$K_3 = hf\left[x+\frac{h}{2}, y+\frac{K_2}{2}\right]$$

$$K_4 = hf[x+h, y+K_3]$$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

Now $y_1 = y_0 + \Delta y$

I. Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 2$. Compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by RK method of fourth order.

Soln.

Given $y' = x^3 + y$; $f(x, y) = x^3 + y$

Here $x_0 = 0$, $y_0 = 2$, $h = x_1 - x_0 = 0.2 - 0 = 0.2$

Now $K_1 = hf(x_0, y_0) = 0.2\, f(0, 2)$

$= 0.2(0^3+2)$

$= 0.2(2)$

$= 0.4$

$K_2 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right] = 0.2f\left[0+\frac{0.2}{2}, 2+\frac{0.4}{2}\right]$

$= 0.2f[0.1, 2.2]$

$= 0.2\left[(0.1)^3+2.2\right]$

$= 0.4402$

$K_3 = hf\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right]$

$= 0.2f\left[0+\frac{0.2}{2}, 2+\frac{0.4402}{2}\right]$

$= 0.2f[0.1, 2.2201]$

$= 0.2\left[(0.1)^3+2.2201\right]$

$= 0.4442$

$K_4 = hf[x_0+h, y_0+K_3] = 0.2f[0+0.2, 2+0.4442]$

$= 0.2f[0.2, 2.4442]$

$= 0.2\left[(0.2)^3+2.4442\right]$

$= 0.4904$

$\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= \frac{1}{6}[0.4+2(0.4402)+2(0.4442)+0.4904]$

$= 0.4432$

$y_1 = y_0+\Delta y$

$= 2+0.4432$

$= 2.4432$

Now $K_1 = hf(x_1, y_1) = 0.2\, f(0.2, 2.4432)$

$= 0.2\left[(0.2)^3+2.4432\right]$

$= 0.4902$

$K_2 = h\,f\left[x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right]$

$= 0.2\,f\left[0.2+\frac{0.2}{2}, 2.4432+\frac{0.4902}{2}\right]$

$= 0.2\,f(0.3, 2.6883)$

$= 0.2\left[(0.3)^3 + 2.6883\right]$

$K_2 = 0.5431$

$K_3 = h\,f\left[x_1+\frac{h}{2}, y_1+\frac{K_2}{2}\right]$

$= 0.2\,f\left[0.2+\frac{0.2}{2}, 2.4432+\frac{0.5431}{2}\right]$

$= 0.2\,f(0.3, 2.7148)$

$= 0.2\left[(0.3)^3 + 2.7148\right]$

$K_3 = 0.5484$

$K_4 = h\,f\left[x_1+h, y_1+K_3\right]$

$= 0.2\,f\left[0.2+0.2, 2.4432+0.5484\right]$

$= 0.2\,f\left[0.4, 2.9916\right]$

$= 0.2\left[(0.4)^3 + 2.9916\right]$

$K_4 = 0.6111$

$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$

$= \frac{1}{6}\left[0.4902 + 2(0.5431) + 2(0.5484) + 0.6111\right]$

$\Delta y = 0.5474$

$y_2 = y_1 + \Delta y = 2.4432 + 0.5474$

$= 2.9906$

Here $x_2 = 0.4$, $y_2 = 2.9906$ and $h = x_2 - x_1$
$= 0.4 - 0.2 = 0.2$

Now, $K_1 = h\,f[x_2, y_2] = 0.2\,f(0.4, 2.9906)$

$= 0.2\left[(0.4)^3 + 2.9906\right]$

$K_1 = 0.6109$

$K_2 = h\,f\left[x_2+\frac{h}{2}, y_2+\frac{K_1}{2}\right]$

$= 0.2\,f\left[0.4+\frac{0.2}{2}, 2.9906+\frac{0.6109}{2}\right]$

$= 0.2\,f\left[0.5, 3.2961\right] = 0.2\left[(0.5)^3 + 3.2961\right]$

$K_2 = 0.6842$

$K_3 = hF\left[x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right]$

$= 0.2\ F\left[0.4 + \frac{0.2}{2}, 2.9906 + \frac{0.6842}{2}\right]$

$= 0.2\ F[0.5, 3.3327]$

$= 0.2\ [(0.5)^3 + 3.3327]$

$K_3 = 0.6915$

$K_4 = hF[x_2 + h, y_2 + K_3]$

$= 0.2\ F[0.4 + 0.2, 2.9906 + 0.6915]$

$= 0.2\ F[0.6, 3.6821]$

$= 0.2\ [(0.6)^3 + 3.6821]$

$K_4 = 0.7796$

$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}(0.6109 + 2(0.6842) + 2(0.6915) + 0.7796)$

$= \frac{1}{6}(4.1419)$

$\Delta y = 0.6903$

$y_3 = y_2 + \Delta y = 2.9906 + 0.6903$

$= 3.6809$

2). Using R-K method of 4^{th} order,

Solve $y' = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$

Soln.

Given $y' = \frac{y^2 - x^2}{y^2 + x^2}$; $F(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

Here $x_0 = 0$, $y_0 = 1$, $h = x_1 - x_0 = 0.2 - 0 = 0.2$

$x_1 = 0.2$, $y_1 = ?$

Now, $K_1 = hF(x_0, y_0)$

$= 0.2\ F(0, 1) = 0.2\left(\frac{1-0}{1+0}\right) = 0.2$

$$K_2 = hF\left[x_0+\frac{h}{2},\ y_0+\frac{K_1}{2}\right]$$
$$=0.2\,F\left[0+\frac{0.2}{2},\ 1+\frac{0.2}{2}\right]$$
$$=0.2\,F[0.1,\ 1.1]$$
$$=0.2\left[\frac{(1.1)^2-(0.1)^2}{(1.1)^2+(0.1)^2}\right]$$
$$=0.2\left[\frac{1.2}{1.22}\right]$$
$$=0.1967$$

$$K_3 = hF\left[x_0+\frac{h}{2},\ y_0+\frac{K_2}{2}\right]$$
$$=0.2\,F\left[0+\frac{0.2}{2},\ 1+\frac{0.1967}{2}\right]$$
$$=0.2\,F[0.1,\ 1.0984]$$
$$=0.2\left[\frac{(1.0984)^2-(0.1)^2}{(1.0984)^2+(0.1)^2}\right]$$
$$=0.2(0.9836)$$
$$K_3 = 0.1967$$

$$K_4 = hF(x_0+h,\ y_0+K_3)$$
$$=0.2\,F(0+0.2,\ 1+0.1967)$$
$$=0.2\,F(0.2,\ 1.1967)$$
$$=0.2\left[\frac{(1.1967)^2-(0.2)^2}{(1.1967)^2+(0.2)^2}\right]$$
$$=0.2(0.9457)$$
$$K_4 = 0.1891$$

$$\Delta y = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$$
$$=\frac{1}{6}\left[0.2+2(0.1967)+2(0.1967)+0.1891\right]$$
$$=0.1960$$

$$y_1 = y_0+\Delta y = 1+0.196 = 1.196$$

3. find $y(0.8)$ given that $y' = y - x^2$, $y(0.6) = 1.7379$ by using RK method of 4th order. Take $h = 0.1$.

Soln.

Given $y' = y - x^2$; $f(x, y) = y - x^2$

Here $x_0 = 0.6$, $y_0 = 1.7379$ $h = 0.1$

$x_1 = 0.7$, $y_1 = ?$

$x_2 = 0.8$, $y_2 = ?$,

Now, $K_1 = hf[x_0, y_0] = 0.1 f[0.6, 1.7379]$

$= 0.1[1.7379 - (0.6)^2]$

$= 0.1 (1.3779)$

$K_1 = 0.1378$

$K_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right]$

$= 0.1 f\left[0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.1378}{2}\right]$

$= 0.1 f[0.65, 1.8068]$

$= 0.1(1.8068 - (0.65)^2)$

$= 0.1384$

$K_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right]$

$= 0.1 f\left[0.6 + \frac{0.1}{2}, 1.7379 + \frac{0.1384}{2}\right]$

$= 0.1 f[0.65, 1.8071]$

$= 0.1 (1.8071 - (0.65)^2)$

$= 0.1385$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.1 f(0.6 + 0.1, 1.7379 + 0.1385)$

$= 0.1 f(0.7, 1.8764)$

$= 0.1(1.8764 - (0.7)^2)$

$= 0.1386$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}(0.1378 + 2(0.1384) + 2(0.1385) + 0.138$$

$$\Delta y = 0.1384$$

$$y_1 = y_0 + \Delta y = 1.7379 + 0.1384$$

$$= 1.8763$$

Here $x_1 = 0.7, \quad y_1 = 1.8763$

$$K_1 = hf(x_1, y_1) = 0.1 f(0.7, 1.8763)$$

$$= 0.1(1.8763 - (0.7)^2)$$

$$= 0.1386$$

$$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$$

$$= 0.1 f(0.7 + \frac{0.1}{2}, 1.8763 + \frac{0.1386}{2})$$

$$= 0.1 f(0.75, 1.9456)$$

$$= 0.1(1.9456 - (0.75)^2)$$

$$= 0.1383$$

$$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2})$$

$$= 0.1 f(0.7 + \frac{0.1}{2}, 1.8763 + \frac{0.1383}{2})$$

$$= 0.1 f(0.75, 1.9455)$$

$$= 0.1(1.9455 - (0.75)^2)$$

$$= 0.1383$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$= 0.1 f(0.7 + 0.1, 1.8763 + 0.1383)$$

$$= 0.1 f(0.8, 2.0146)$$

$$= 0.1(2.0146 - (0.8)^2)$$

$$= 0.1375$$

$$\Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}(0.1386 + 2(0.1383) + 2(0.1383) + 0.1375)$$

$\Delta y = 0.1382$

$y_2 = y_1 + \Delta y = 1.8763 + 0.1382$

$y_2 = 2.0145$.

Runge Kutta method for 2^{nd} order differential Equations:

find the soln. of $y'' = f(x, y, y')$

given $y(x_0) = y_0$, $y'(x_0) = y_0'$.

Now set $y' = z$ and $y'' = z'$

Hence, differential eqn. reduces to

$$\frac{dy}{dx} = y' = z$$

and $$\frac{dz}{dx} = z' = y'' = f(x, y, y') = f(x, y, z)$$

$\therefore \frac{dy}{dx} = z$ and $\frac{dz}{dx} = f(x, y, z)$ are simultaneous eqns. where $f_1(x, y, z) = z$ and $f_2(x, y, z) = f(x, y, z)$ given. Also $y(0)$ and $z(0)$ are given.

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① Solving the system of differential equations $\frac{dy}{dx} = xz + 1$, $\frac{dz}{dx} = -xy$, for $x = 0.3$ using fourth order R_K method, the initial values are $x = 0$, $y = 0$, $z = 1$.

Soln.

Given $x_0 = 0$, $y_0 = 0$, $z_0 = 1$, $h = x_1 - x_0$

$= 0.3 - 0 = 0.3$

$x_1 = 0.3$, $y_1 = ?$

$f_1(x,y,z) = xz+1$

$K_1 = hf_1(x_0, y_0, z_0)$

$= 0.3(x_0 z_0 + 1)$

$= 0.3(0+1) = 0.3$

$K_2 = hf_1\left[x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{l_1}{2}\right]$

$= 0.3f_1\left[0+\frac{0.3}{2}, 0+\frac{0.3}{2}, 1+\frac{0}{2}\right]$

$= 0.3f_1[0.15, 0.15, 1]$

$= 0.3(0.15(1)+1)$

$= 0.345$

$K_3 = hf_1\left[x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right]$

$= 0.3f_1\left[0.15, 0+\frac{0.345}{2}, 1-\frac{0.007}{2}\right]$

$= 0.3f_1[0.15, 0.1725, 0.9965]$

$= 0.3[(0.15)(0.9965)+1]$

$= 0.3448$

$K_4 = hf_1[x_0+h, y_0+K_3, z_0+l_3]$

$= 0.3f_1[0+0.3, 0+0.3448, 1-0.0078]$

$= 0.3f_1(0.3, 0.3448, 0.9922)$

$= 0.3[0.3(0.9922)+1]$

$= 0.3893$

$\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= \frac{1}{6}[0.3+2(0.345)+2(0.3448)+0.3893]$

$= 0.3448$

$y_1 = y_0 + \Delta y$

$= 0 + 0.34482$

$= 0.34482$

$f_2(x,y,z) = -xy$

$l_1 = hf_2(x_0, y_0, z_0)$

$= 0.3(-x_0 y_0)$

$= -0.3(0) = 0$

$l_2 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}, z_0+\frac{l_1}{2}\right)$

$= 0.3f_2\left(0+\frac{0.3}{2}, 0+\frac{0.3}{2}, 1+\frac{0}{2}\right)$

$= 0.3f_2(0.15, 0.15, 1)$

$= 0.3(-0.15\times 0.15)$

$= -0.007$

$l_3 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}, z_0+\frac{l_2}{2}\right)$

$= 0.3f_2\left[0.15, 0.1725, 1-\frac{0.007}{2}\right]$

$= 0.3f_2[0.15, 0.1725, 0.9965]$

$= 0.3(-0.15\times 0.1725)$

$= -0.0078$

$l_4 = hf_2(x_0+h, y_0+K_3, z_0+l_3)$

$= 0.3f_2(0.3, 0.3448, 0.9922)$

$= 0.3(-0.3\times 0.3448)$

$= -0.031032$

$\Delta z = \frac{1}{6}[l_1+2l_2+2l_3+l_4]$

$= \frac{1}{6}[0+2(-0.007)+2(-0.0078)+(-0.031032)]$

$= -0.01011$

$z_1 = z_0 + \Delta z$

$= 1 - 0.01011$

$= 0.9899$

1. Consider the 2^{nd} order IVP $y'' - 2y' + 2y = e^{2t} \sin t$, $y(0) = -0.4$ and $y'(0) = -0.6$ using 4^{th} order R_K method, find $y(0.2)$.

Soln.

Let $t = x$.

$y'' - 2y' + 2y = e^{2x} \sin x$

$y(0) = -0.4$, $y'(0) = -0.6$, $h = 0.2$

Setting $y' = z$, the eqn. becomes,

$z' = 2z - 2y + e^{2x} \sin x$

$f_1(x, y, z) = \frac{dy}{dx} = z$, $f_2(x, y, z) = \frac{dz}{dx} = 2z - 2y + e^{2x} \sin x$

Given $y_0 = -0.4$, $z_0 = y_0' = -0.6$, $x_0 = 0$

$k_1 = h f_1(x_0, y_0, z_0)$

$= 0.2(z_0)$

$= 0.2(-0.6)$

$k_1 = -0.12$

$l_1 = h f_2(x_0, y_0, z_0)$

$= 0.2[2z_0 - 2y_0 + e^{2x_0} \sin x_0]$

$= 0.2[2(-0.6) - 2(-0.4) + e^{2(0)} \sin(0)]$

$= 0.2[-0.12 + 0.8]$

$l_1 = -0.08$

$k_2 = h f_1 \left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= 0.2 f_1 \left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 + \frac{0.08}{2}\right]$

$= 0.2 f_1 [0.1, -0.46, -0.64]$

$= 0.2(-0.64)$

$= -0.128$

$l_2 = h f_2 \left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right]$

$= 0.2 f_2 \left[0 + \frac{0.2}{2}, -0.4 - \frac{0.12}{2}, -0.6 - \frac{0.08}{2}\right]$

$= 0.2 f_2 [0.1, -0.46, -0.64]$

$= 0.2(2(-0.64) - 2(-0.46) + e^{2(0.1)} \sin(0.1))$

0.2381

$= -0.0476$

$k_3 = h f_1 \left(x_0 + h/2, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.2 f_1 \left(0 + \frac{0.2}{2}, -0.4 - \frac{0.1280}{2}, -0.6 - \frac{0.0476}{2}\right)$

$= 0.2 f_1 (0.1, -0.464, -0.6238)$

$= 0.2(-0.6238) = -0.1248$

$l_3 = h f_2 \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= 0.2 f_2 \left(0 + \frac{0.2}{2}, -0.4 + \frac{0.128}{2}, -0.6 - \frac{0.0476}{2}\right)$

$= 0.2 f_2 (0.1, -0.464, -0.6238)$

$= 0.2 [2(-0.6238) - 2(-0.464) + e^{2(0.1)} \sin(0.1)]$

$= -0.0395$

$$K_4 = hf_1(x_0+h, y_0+K_3, z_0+l_3)$$
$$= 0.2f_1(0+0.2, -0.4-0.1248, -0.6-0.0395)$$
$$= 0.2f_1(0.2, -0.5248, -0.6395)$$
$$= 0.2(-0.6395)$$
$$= -0.1279$$

$$l_4 = hf_2[x_0+h, y_0+K_3, z_0+l_3]$$
$$= 0.2f_2(0.2, -0.5248, -0.6395)$$
$$= 0.2\left(2(-0.6395) - 2(-0.5248) + e^{2(0.2)}\sin(0.2)\right)$$
$$= 0.0134$$

$$\Delta y = \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$= \frac{1}{6}[-0.12 - 2(0.128) - 2(.1248) - 0.1279]$$
$$= -0.1256$$

$$\Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= \frac{1}{6}[-0.08 - 2(0.0476) - 2(0.0395) + 0.0134]$$
$$= -0.0401$$

$$y_1 = y_0 + \Delta y$$
$$= -0.4 - 0.1256$$
$$y(0.2) = -0.5256$$

$$z_1 = z_0 + \Delta z$$
$$= -0.6 - 0.0401$$
$$= 0.6401$$

Milne's Predictor and Corrector Methods:

Milne's method is a multistep method that first predicts a value for y_{n+1} from 3 past values of the derivatives. The past values are computed using either RK method (or) Taylor series method.

Milne's predictor and corrector formula is,

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$