Unit - 5

Boundary value Problems in Ordinary and Partial Differential Equations

* finite difference methods for solving two point linear boundary value problems
* finite difference techniques for the soln. of two dimensional Laplace's and Poisson's equations on rectangular domain
* one dimensional wave eqn. by explicit method.

Boundary Value Problem (BVP)

The differential equation together with the boundary conditions is called a boundary value problem.

Finite difference method:

This method is used to solve ODEs - Boundary value problems (BVP).

The general linear two point BVP is,

$$y'' = P(x)y' + q(x)y + r(x), \quad a \le x \le b$$

with boundary conditions $y(a) = \alpha$ & $y(b) = \beta$.

* To solve this problem by using finite differences, we divide the interval $[a, b]$ into n subintervals so that $h = \frac{b-a}{n}$.

* To approximate the function $y(x)$ at points $x_1 = a+h$, $x_2 = a+2h$, $x_3 = a+3h$, ... $x_n = a+(n-1)h$ we use the following central difference formula.

$$y''(x_i) \simeq \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$y'(x_i) \simeq \frac{y_{i+1} - y_{i-1}}{2h}$$

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1. Solve $y'' + y + 1 = 0$ with the boundary conditions $y = 0$ when $x = 0$ and $y = 0$ when $x = 1$ by finite difference method with $n = 4$.

Soln.

Since $[0, 1]$ is divided into 4 sub intervals i.e., $n = 4$, we have $h = \frac{b-a}{n} = \frac{1-0}{4}$

$$h = 1/4$$

Given $y'' + y + 1 = 0 \rightarrow (1)$

Using the central difference formula,

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \rightarrow (2)$$

Subs. (2) in (1),

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i + 1 = 0 \qquad \because y = y(x_i) = y_i$$

$$y_{i-1} - 2y_i + y_{i+1} + h^2 y_i + h^2 = 0$$

$$y_{i-1} - 2y_i + y_{i+1} + \frac{1}{16} y_i + \frac{1}{16} = 0$$

$$16y_{i-1} - 32y_i + 16y_{i+1} + y_i + 1 = 0$$

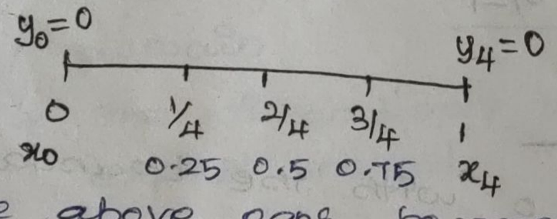
$$16y_{i-1} - 31y_i + 16y_{i+1} + 1 = 0$$

when $i=1$, $16y_0 - 31y_1 + 16y_2 = -1$

$i=2$, $16y_1 - 31y_2 + 16y_3 = -1$

$i=3$, $16y_2 - 31y_3 + 16y_4 = -1$

By boundary conditions $y_0 = 0$ & $y_4 = 0$



The above eqns. becomes,

$$-31y_1 + 16y_2 = -1 \rightarrow (3)$$

$$16y_1 - 31y_2 + 16y_3 = -1 \rightarrow (4)$$

$$16y_2 - 31y_3 = -1 \rightarrow (5)$$

Solve (3), (4) and (5).

(3) − (5) $\Rightarrow$

$$\begin{array}{rrrr} -31y_1 & +16y_2 & +0y_3 & = -1 \\ 0y_1 & +16y_2 & -31y_3 & = -1 \\ (-) & (-) & (+) & (+) \\ \hline -31y_1 & & +31y_3 & = 0 \end{array}$$

$$-31(y_1 - y_3) = 0$$

$\Rightarrow y_1 - y_3 = 0$

$\Rightarrow y_1 = y_3$

Subs. $y_1 = y_3$ in (4),

$16y_3 - 31y_2 + 16y_3 = -1$

$-31y_2 + 32y_3 = -1 \rightarrow (6)$

$\therefore 31y_2 - 32y_3 = 1$

Solve (5) & (6),

$16y_2 - 31y_3 = -1$

$-31y_2 + 32y_3 = 1$

$(5)\times 31 \Rightarrow 496y_2 - 961y_3 = -31$

$(6)\times 16 \Rightarrow -496y_2 + 512y_3 = -16$

$-449y_3 = -47$

$y_3 = 0.1047$

Subs. y_3 in (5),

$16y_2 - 31(0.1047) = -1$

$16y_2 = -1 + 3.2457$

$= 2.2457$

$y_2 = 0.1404$

$\because y_1 = y_3 = 0.1047$

Hence $y(0.25) = y(0.75) = 0.1047$

$y(0.5) = 0.1404$

2) Solve by finite difference method, the boundary value problem $y''(x) - y(x) = 2$ where $y(0) = 0$ and $y(1) = 1$ taking $h = 1/4$.

Soln.

Given $h = \frac{1}{4}$ and $y''(x) - y(x) = 2 \rightarrow (1)$

Using the central difference formula,

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \rightarrow (2)$$

Sub.. (2) in (1),

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - y_i = 2$$

$$16(y_{i-1} - 2y_i + y_{i+1}) - y_i = 2$$

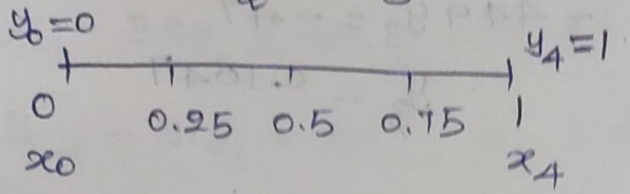
$$16y_{i-1} - 32y_i + 16y_{i+1} - y_i = 2$$

$$16y_{i-1} - 33y_i + 16y_{i+1} = 2$$

when $i=1$, $16y_0 - 33y_1 + 16y_2 = 2$

$i=2$, $16y_1 - 33y_2 + 16y_3 = 2$

$i=3$, $16y_2 - 33y_3 + 16y_4 = 2$



By boundary conditions, $y_0 = 0$ & $y_4 = 1$.
The above eqns. becomes,

$$-33y_1 + 16y_2 = 2 \rightarrow (3)$$

$$16y_1 - 33y_2 + 16y_3 = 2 \rightarrow (4)$$

$$16y_2 - 33y_3 + 16 = 2 \Rightarrow 16y_2 - 33y_3 = 2 - 16 = -14 \rightarrow (5)$$

Solve (3), (4) & (5)

$(3) - (5) \Rightarrow$

$$-33y_1 + 16y_2 + 0y_3 = 2$$
$$0y_1 + 16y_2 - 33y_3 = -14$$
$$(-)\quad(-)\quad(+)\quad(+)$$
$$-33y_1 + 33y_3 = 16 \rightarrow (6)$$

$33 \times (3) + 16 \times (4)$

$$\Rightarrow 1089y_1 + 528y_2 + 0y_3 = 66$$
$$256y_1 - 528y_2 + 256y_3 = 32$$

$$1345y_1 + 256y_3 = 98 \rightarrow (7)$$

Solve (6) & (7),

$256 \times (6) - 33 \times (7)$

$$-8448 y_1 + 8448 y_3 = 4096$$

$$\Rightarrow \quad \underset{(-)}{44385 y_1} \underset{(-)}{+ 8448 y_3} = \underset{(-)}{3234}$$

$$-52833 y_1 = 862$$

$$y_1 = -0.0163$$

$$(6) \Rightarrow \quad -33 y_1 + 33 y_3 = 16$$

$$-33(-0.0163) + 33 y_3 = 16$$

$$33 y_3 = 16 - 0.5384$$

$$= 15.4616$$

$$y_3 = 0.4685$$

$$(3) \Rightarrow \quad -33 y_1 + 16 y_2 = 2$$

$$-33(-0.0163) + 16 y_2 = 2$$

$$16 y_2 = 2 - 0.5379$$

$$= 1.4621$$

$$y_2 = 0.0914$$

Hence $y(0.25) = -0.0163$

$y(0.5) = 0.0914$

$y(0.75) = 0.4685$

5. Using finite difference method, find $y(0.25)$, $y(0.5)$, $y(0.75)$ satisfying the diff. eqn. subject to the boundary conditions

$$\frac{d^2 y}{dx^2} + y = x$$

$y(0) = 0$, $y(1) = 1$.

Soln.

Given $y'' + y = x$ $\quad\quad$ and $h = 0.25$ $\quad$ (1)

Here $y(0) = 0$, $y(1) = 1$

$x_0 = 0$, $y_0 = 0$	$x_2 = 0.5$, $y_2 = ?$
$x_1 = 0.25$, $y_1 = ?$	$x_3 = 0.75$, $y_3 = ?$
	$x_4 = 1$, $y_4 = 1$

Using the central difference formula

$$y'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$$

$$(1) \Rightarrow \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + y_i = x_i$$

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{1/16} + y_i = x_i$$

$$16(y_{i-1} - 2y_i + y_{i+1}) + y_i = x_i$$

$$16y_{i-1} - 32y_i + 16y_{i+1} + y_i = x_i$$

$$16y_{i-1} - 31y_i + 16y_{i+1} = x_i$$

Put $i=1$,

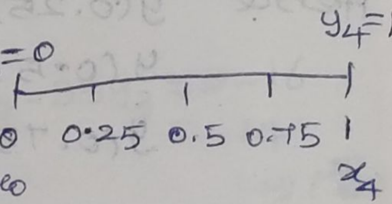
$$16y_0 - 31y_1 + 16y_2 = x_1 = 0.25$$

Put $i=2$, $16y_1 - 31y_2 + 16y_3 = x_2 = 0.5$

Put $i=3$, $16y_2 - 31y_3 + 16y_4 = x_3 = 0.75$

By boundary conditions,

$y_0 = 0$, $y_4 = 1$



The above eqns becomes,

$$-31y_1 + 16y_2 = 0.25 \rightarrow (2)$$

$$16y_1 - 31y_2 + 16y_3 = 0.5 \rightarrow (3)$$

$$16y_2 - 31y_3 = -15.25 \rightarrow (4)$$

$$\begin{array}{r} -16 \\ 0.75 \\ \hline 15.25 \end{array}$$

Solve (2), (3) & (4),

$$(2)-(4) \Rightarrow \begin{array}{l} -31y_1 + 16y_2 + 0y_3 = 0.25 \\ 0y_1 + 16y_2 - 31y_3 = -15.25 \\ (-)\quad\ (-)\quad\ (+)\quad\quad (+) \\ \hline -31y_1 + 31y_3 = 15.5 \rightarrow (5) \end{array}$$

$$31 \times (2) + 16 \times (3) \Rightarrow \begin{array}{l} -961y_1 + 496y_2 + 0y_3 = 7.75 \\ 256y_1 - 496y_2 + 256y_3 = 8 \\ \hline -705y_1 + 256y_3 = 15.75 \rightarrow (6) \end{array}$$

Solve (5) & (6)

$705 \times (15) - 31 \times (6) \Rightarrow$

$-21855 y_1 + 21855 y_3 = 10927.5$

$-21855 y_1 + 7936 y_3 = 488.25$

$\underset{(+)}{} \quad \underset{(-)}{} \quad \underset{(-)}{}$

$13,919 y_3 = 10439.25$

$y_3 = 0.75$

Subs. y_3 in (5), $-31y_1 + 31(0.75) = 15.5$

$31y_1 = 31(0.75) - 15.5$

$= 23.25 - 15.5$

$31y_1 = 7.75$

$y_1 = 0.25$

Sub: y_1 in (2),

$-31(0.25) + 16y_2 = 0.25$

$16y_2 = 0.25 + 31(0.25)$

$= 0.25 + 7.75$

$16y_2 = 8$

$y_2 = 0.5$

$\therefore y_1 = 0.25$

$y_2 = 0.5$

$y_3 = 0.75$

Hw J. Solve, by finite difference method, the boundary value problem $x^2 y''(x) - 2y(x) + x = 0$, $2 < x < 3$ where $y(2) = 0$, $y(3) = 0$, taking $h = 1/4$.