Classification of PDE of 2^{nd} order:

The 2^{nd} order PDE is,

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F(u) = 0$$

(or)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F(u) = 0$$

$B^2 - 4AC = 0 \Rightarrow$ Parabolic

$B^2 - 4AC < 0 \Rightarrow$ Elliptic

$B^2 - 4AC > 0 \Rightarrow$ Hyperbolic.

1]. Classify the PDE $u_{xx} - 2u_{xy} + u_{yy} = 0$

Soln.

Here $A = 1,\ B = -2,\ C = 1$

Now, $B^2 - 4AC = (-2)^2 - 4(1)(1) = 4 - 4 = 0$

$\therefore$ It is parabolic.

2]. Solve $f_{xx} + 2f_{xy} + 4f_{yy} = 0$

Soln.

Here $A = 1,\ B = 2,\ C = 4$

Now $B^2 - 4AC = 4 - 4(1)(4) = -8 < 0$

$\therefore$ It is Elliptic.

3]. Solve $f_{xx} - 2f_{xy} = 0$

Soln.

Here $A = 1,\ B = -2,\ C = 0$

Now $B^2 - 4AC = (-2)^2 - 4(1)(0) = 4 > 0$

$\therefore$ It is hyperbolic.

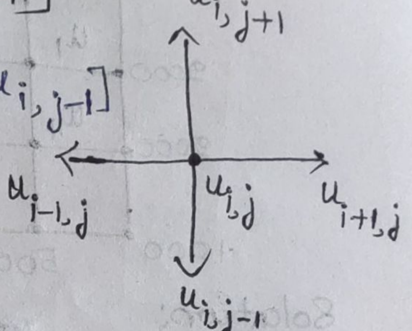
Type-1

Solution of two dimensional laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(laplace equation)}$$

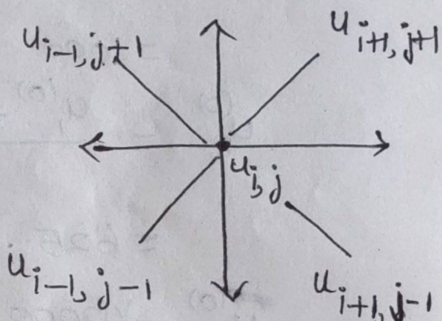
Standard five point formula [SFPF]

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}\right]$$



Diagonal five point formula [DFPF]

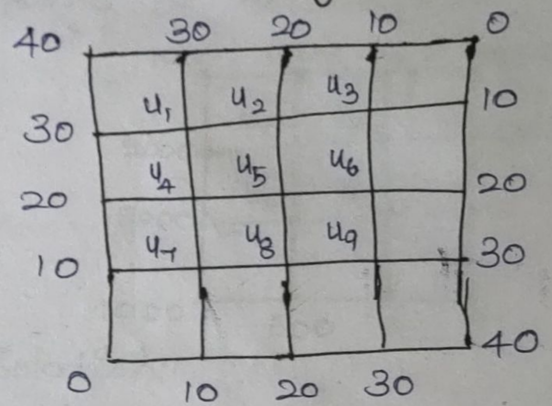
$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1}\right]$$



Liebmann Iteration formula:

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)}\right]$$

II. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ at the nine mesh points of the square given below. The values of u at the boundary are specified in the figure.



Solution

we use SFPF & DFPF for initial Iteration

$u_5^{(0)} = \frac{1}{4}[20+20+20+20]$. (By Standard formula)

$= 20$

By Diagonal formula,

$u_1^{(0)} = \frac{1}{4}(40+20+20+u_5^{(0)})$

$= \frac{1}{4}(40+20+20+20)$

$= 25$

$u_3^{(0)} = \frac{1}{4}(0+30+20+u_5^{(0)})$

$= \frac{1}{4}(0+30+20+20)$

$= 15$

$u_7^{(0)} = \frac{1}{4}(20+0+20+u_5^{(0)})$

$= \frac{1}{4}(20+0+20+20)$

$= 15$

$u_9^{(0)} = \frac{1}{4}(20+40+20+u_5^{(0)})$

$= \frac{1}{4}(20+40+20+20)$

$= 25$

By Standard formula,

$u_2^{(0)} = \frac{1}{4}(20+u_3^{(0)}+u_5^{(0)}+u_1^{(0)})$

$= \frac{1}{4}(20+15+20+25)$

$= 20$

$u_4^{(0)} = \frac{1}{4}(u_1^{(0)} + u_7^{(0)} + 20 + 20)$

$= \frac{1}{4}(25+15+20+20)$

$= 20$

$u_6^{(0)} = \frac{1}{4}(u_3^{(0)} + u_9^{(0)} + 20 + 20)$

$= \frac{1}{4}(15+25+20+20)$

$= 20$

$u_8^{(0)} = \frac{1}{4}(u_7^{(0)} + u_9^{(0)} + 20 + 20)$

$= \frac{1}{4}(15+25+20+20)$

$= 20$

First Iteration :

By standard formula,

$u_1^{(1)} = \frac{1}{4}(u_2^{(0)} + u_4^{(0)} + 30 + 30)$

$= \frac{1}{4}(20+20+30+30)$

$= 25$

$u_2^{(1)} = \frac{1}{4}[u_1^{(1)} + u_3^{(0)} + u_5^{(0)} + 20]$

$= \frac{1}{4}(25+15+20+20)$

$= 20$

$u_3^{(1)} = \frac{1}{4}(u_2^{(1)} + u_6^{(0)} + 10 + 10)$

$= \frac{1}{4}(20+20+10+10)$

$= 15$

$u_4^{(1)} = \frac{1}{4}(u_1^{(1)} + u_5^{(0)} + u_7^{(0)} + 20)$

$= \frac{1}{4}(25+20+15+20)$

$= 20$

$$u_5^{(1)} = \frac{1}{4}\left(u_2^{(1)} + u_4^{(1)} + u_6^{(0)} + u_8^{(0)}\right)$$

$$= \frac{1}{4}(20+20+20+20)$$

$$= 20$$

$$u_6^{(1)} = \frac{1}{4}\left(u_3^{(1)} + u_5^{(1)} + u_9^{(0)} + 20\right)$$

$$= \frac{1}{4}(15+20+25+20)$$

$$= 20$$

$$u_7^{(1)} = \frac{1}{4}\left(u_4^{(1)} + u_8^{(0)} + 10 + 10\right) = \frac{1}{4}(20+20+10+10)$$

$$= 15$$

$$u_8^{(1)} = \frac{1}{4}\left(u_7^{(1)} + u_5^{(1)} + u_9^{(0)} + 20\right) = \frac{1}{4}(15+20+25+20)$$

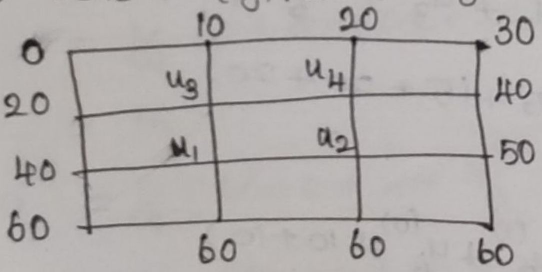
$$= 20$$

$$u_9^{(1)} = \frac{1}{4}\left(u_8^{(1)} + u_6^{(1)} + 30 + 30\right) = \frac{1}{4}(20+20+30+30)$$

$$= 25$$

$\therefore$ $u_1 = 25$, $u_2 = 20$, $u_3 = 15$, $u_4 = 20$, $u_5 = 20$
$u_6 = 20$, $u_7 = 15$, $u_8 = 20$, $u_9 = 25$.

2]. Solve $\nabla^2 u = 0$, the boundary conditions are given below (given only three iteration)



Soln.

Take $u_4 = 0$

Initial Iteration:

By Diagonal formula,

$$u_1^{(0)} = \frac{1}{4}[u_4 + 60 + 60 + 20]$$

$$= \frac{1}{4}[0 + 60 + 60 + 20]$$

$$= 35$$

By Standard formula

$$u_2^{(0)} = \frac{1}{4}[u_4 + 50 + 60 + u_1^{(0)}]$$
$$= \frac{1}{4}[0 + 50 + 60 + 35]$$
$$= 36.25$$

$$u_3^{(0)} = \frac{1}{4}[u_4 + u_1^{(0)} + 10 + 20]$$
$$= \frac{1}{4}[0 + 35 + 10 + 20]$$
$$= 16.25$$

$$u_4^{(0)} = \frac{1}{4}[20 + 40 + u_2^{(0)} + u_3^{(0)}]$$
$$= \frac{1}{4}[20 + 40 + 36.25 + 16.25]$$
$$= 28.125$$

First Iteration:

$$u_1^{(1)} = \frac{1}{4}[u_3^{(0)} + u_2^{(0)} + 60 + 40]$$
$$= \frac{1}{4}[16.25 + 36.25 + 60 + 40] = 38.125$$

$$u_2^{(1)} = \frac{1}{4}(u_4^{(0)} + 50 + 60 + u_1^{(1)}) = \frac{1}{4}(28.125 + 50 + 60 + 38.125)$$
$$= 44.063$$

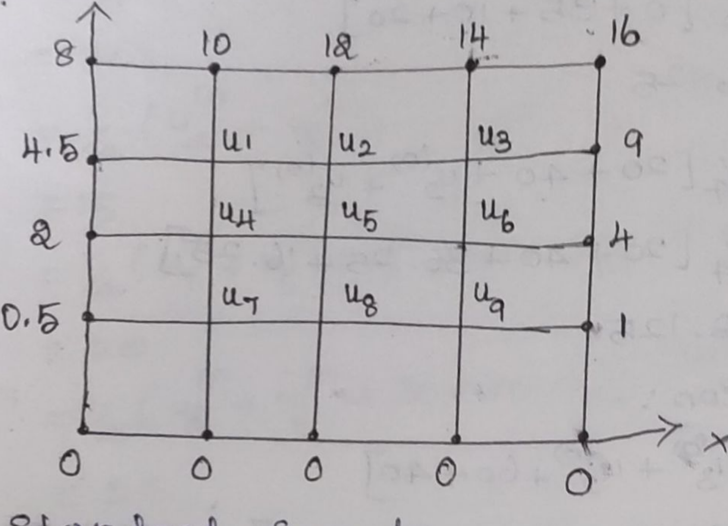
$$u_3^{(1)} = \frac{1}{4}(u_4^{(0)} + u_1^{(1)} + 10 + 20) = \frac{1}{4}(28.125 + 38.125 + 10 + 20)$$
$$= 24.063$$

$$u_4^{(1)} = \frac{1}{4}(20 + 40 + u_2^{(1)} + u_3^{(1)}) = \frac{1}{4}(20 + 40 + 44.063 + 24.063)$$
$$= 32.032$$

Iteration	u_1	u_2	u_3	u_4
1st	38.125	44.063	24.063	32.032
2nd	42.032	46.016	26.016	33.008
3rd	43.008	46.504	26.504	33.252
4th	43.252	46.565	26.565	33.283
5th	43.283	46.642	26.642	33.321

3. Given that $u(x, y)$ satisfies the equation $\nabla^2 u = 0$ and the boundary conditions $u(x, 0) = 0$, $u(x, 4) = 8 + 2x$, $u(0, y) = \frac{1}{2} y^2$ and $u(4, y) = y^2$, find the values of $u(i, j)$, $i = 1, 2, 3$; $j = 1, 2, 3$, correct to two places of decimals, by Liebmann's Iteration method.

Soln.



By Standard formula,

$$u_5^{(0)} = \frac{1}{4}(12 + 4 + 0 + 2)$$

$$= 4.5$$

By diagonal formula

$$u_1^{(0)} = \frac{1}{4}(8 + 12 + 2 + u_5^{(0)}) = \frac{1}{4}(8 + 12 + 2 + 4.5)$$

$$= 6.63$$

$$u_3^{(0)} = \frac{1}{4}(12 + 16 + 4 + u_5^{(0)}) = \frac{1}{4}(12 + 16 + 4 + 4.5)$$

$$= 9.13$$

$$u_7^{(0)} = \frac{1}{4}(0 + 0 + u_5^{(0)} + 2) = \frac{1}{4}(0 + 0 + 4.5 + 2)$$

$$= 1.63$$

$$u_9^{(0)} = \frac{1}{4}(0 + 0 + 4 + u_5^{(0)}) = \frac{1}{4}(0 + 0 + 4 + 4.5)$$

$$= 2.13$$

By Standard formula,

$$u_2^{(0)} = \frac{1}{4}(12 + u_1^{(0)} + u_3^{(0)} + u_5^{(0)}) = \frac{1}{4}(12 + 6.63 + 9.13 + 4.5)$$

8.07

$u_4^{(0)} = \frac{1}{4}(2 + u_1^{(0)} + u_5^{(0)} + u_7^{(0)}) = \frac{1}{4}(2 + 6.63 + 4.5 + 1.63)$

$= 3.69$

$u_6^{(0)} = \frac{1}{4}(4 + u_3^{(0)} + u_5^{(0)} + u_9^{(0)}) = \frac{1}{4}(4 + 9.13 + 4.5 + 2.13)$

$= 4.95$

$u_8^{(0)} = \frac{1}{4}(0 + u_5^{(0)} + u_7^{(0)} + u_9^{(0)}) = \frac{1}{4}(0 + 4.5 + 1.63 + 2.13)$

$= 2.07$

first Iteration: By standard formula,

$u_1^{(1)} = \frac{1}{4}(10 + 4.5 + u_2^{(0)} + u_4^{(0)}) = \frac{1}{4}(10 + 4.5 + 8.07 + 3.69)$

$= 6.57$

$u_2^{(1)} = \frac{1}{4}(12 + u_1^{(1)} + u_3^{(0)} + u_5^{(0)}) = \frac{1}{4}(12 + 6.57 + 9.13 + 4.5)$

$= 8.05$

$u_3^{(1)} = \frac{1}{4}(14 + 9 + u_2^{(1)} + u_6^{(1)}) = \frac{1}{4}(14 + 9 + 8.05 + 4.95)$

$= 9$

$u_4^{(1)} = \frac{1}{4}(2 + u_1^{(1)} + u_5^{(0)} + u_7^{(0)}) = \frac{1}{4}(2 + 6.57 + 4.5 + 1.63)$

$= 3.68$

$u_5^{(1)} = \frac{1}{4}(u_2^{(1)} + u_4^{(1)} + u_8^{(0)} + u_6^{(0)}) = \frac{1}{4}(8.05 + 3.68 + 2.07 + 4.95)$

$= 4.69$

$u_6^{(1)} = \frac{1}{4}(4 + u_3^{(1)} + u_5^{(1)} + u_9^{(0)}) = \frac{1}{4}(4 + 9 + 4.69 + 2.13)$

$\doteq 4.96$

$u_7^{(1)} = \frac{1}{4}(0 + 0.5 + u_4^{(1)} + u_8^{(0)}) = \frac{1}{4}(0 + 0.5 + 3.68 + 2.07)$

$= 1.56$

$u_8^{(1)} = \frac{1}{4}(0 + u_5^{(1)} + u_7^{(1)} + u_9^{(0)}) = \frac{1}{4}(0 + 4.69 + 1.56 + 2.13)$

$= 2.1$

$u_9^{(1)} = \frac{1}{4}(0 + 1 + u_6^{(1)} + u_8^{(1)}) = \frac{1}{4}(0 + 1 + 4.96 + 2.1)$

$= 2.02$

Iteration	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
1	6.57	8.05	9	3.68	4.69	4.96	1.56	2.10	2.02
2	6.56	8.06	9.01	3.7	4.71	4.94	1.58	2.08	2.01
3	6.57	8.07	9	3.72	4.7	4.93	1.58	2.07	2
4	6.57	8.07	9	3.71	4.7	4.93	1.57	2.07	2

$\therefore u_1 = 6.57,\ u_2 = 8.07,\ u_3 = 9,\ u_4 = 3.71,\ u_5 = 4.7$

$u_6 = 4.93,\ u_7 = 1.57,\ u_8 = 2.07,\ u_9 = 2$

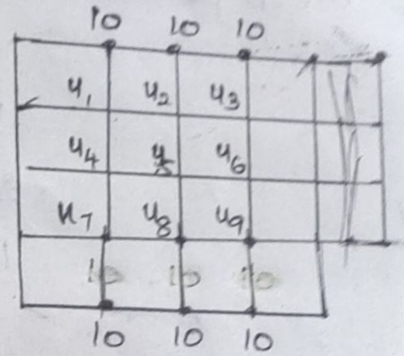
H.W ① Solve the Leibmann's eqn. over the square mesh points satisfy the boundary condition $u(0,y)=0,\ 0 \le y \le 4,\ u(4,y)=12+y,\ 0 \le y \le 4$;
$u(x,0)=3x,\ 0 \le x \le 4,\ u(x,4)=x^2,\ 0 \le x \le 4$

② Solve $u_{xx}+u_{yy}=0,\ 0 \le x,y \le 1$ with $u(0,y)=10=u(1,y)$ and $u(x,0)=20=u(x,1)$. Take $h=0.25$ and apply Liebmann method to decimal accuracy

1]. Symmetric w.r. to x-axis

Here $u_1 = u_7$, $u_2 = u_8$, $u_3 = u_9$

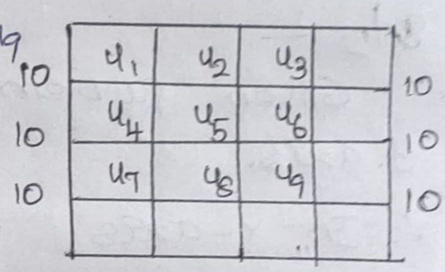
To find $u_1, u_2, u_3, u_4, u_5, u_6$ only



2]. Symmetric w.r. to y-axis.

Here $u_1 = u_3$, $u_4 = u_6$, $u_7 = u_9$

To find $u_1, u_4, u_7, u_2, u_5, u_8$ only.



5]. Symmetric w.r. to x and y axis.

In x-axis,

$u_1 = u_7$

$u_2 = u_8$

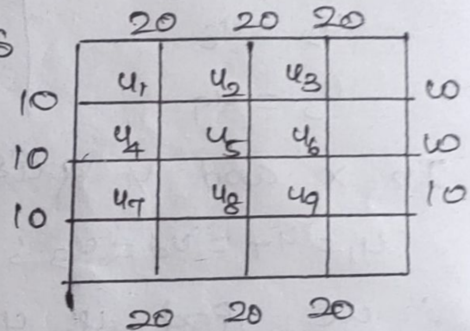
$u_3 = u_9$

In y-axis

$u_1 = u_3$

$u_4 = u_6$

$u_7 = u_9$



$\therefore u_1 = u_7 = u_9 = u_3$

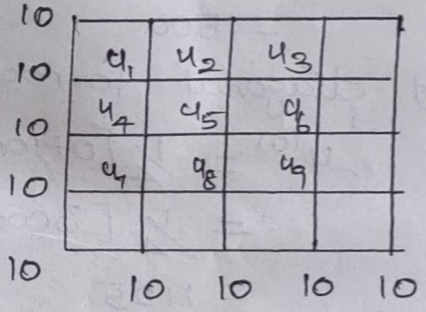
$u_2 = u_8$, $u_4 = u_4$

$\therefore$ we find u_1, u_2, u_4, u_5 only.

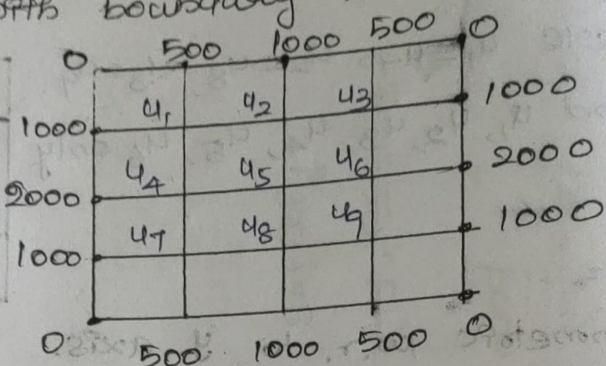
4] Symmetric w.r. to diagonal

Here $u_4 = u_2$, $u_7 = u_3$, $u_8 = u_6$

To find $u_2, u_3, u_6, u_1, u_9, u_5$ only.



I. Solve $\nabla^2 u = 0$ for the following square mesh points with boundary values as shown.



Soln.

Given problem is symmetric w.r.t. to x and y axis.

$\therefore$ In x-axis,

$u_1 = u_7$

$u_2 = u_8$

$u_3 = u_9$

In y-axis

$u_1 = u_3$

$u_4 = u_6$

$u_7 = u_9$

In x and y axis

$u_1 = u_7 = u_9 = u_3$; $u_2 = u_8$, $u_4 = u_6$

$\therefore$ we find u_1, u_2, u_4, u_5 only.

Initial Iteration:

By formula,

$$u_5^{(0)} = \frac{1}{4}(1000 + 2000 + 1000 + 2000)$$

$$= 1500$$

By diagonal formula,

$$u_1^{(0)} = \frac{1}{4}[0 + 1000 + u_5^{(0)} + 2000]$$

$$= \frac{1}{4}[3000 + 1500]$$

$$= 1125$$

By standard formula,

$$u_2^{(0)} = \frac{1}{4}[1000 + u_3^{(0)} + u_5^{(0)} + u_1^{(0)}]$$

$$= \frac{1}{4}[1000 + 1125 + 1500 + 1125]$$

$$= 1187.5$$

$$u_4^{(0)} = \frac{1}{4}[u_1^{(0)} + u_5^{(0)} + u_7^{(0)} + 2000]$$

$= \frac{1}{4}[1125 + 1500 + 1125 + 2000]$

$u_4^{(0)} = 1437.5$

First Iteration: By Standard formula,

$u_1^{(1)} = \frac{1}{4}[500 + u_2^{(0)} + u_4^{(0)} + 1000] = \frac{1}{4}[500 + 1187.5 + 1437.5 + 1000]$

$= 1031.25$

$u_2^{(1)} = \frac{1}{4}[1000 + u_3^{(1)} + u_5^{(0)} + u_1^{(1)}] = \frac{1}{4}[1000 + 1031.25 + 1500 + 1031.25]$

$= 1140.625$

$u_4^{(1)} = \frac{1}{4}[u_1^{(1)} + u_5^{(0)} + u_7^{(1)} + 2000] = \frac{1}{4}[1031.25 + 1500 + 1031.25]$

$= 1390.625$

$u_5^{(1)} = \frac{1}{4}[u_2^{(1)} + u_6^{(1)} + u_8^{(1)} + u_4^{(1)}]$

$= \frac{1}{4}[1140.625 + 1390.625 + 1140.625 + 1390.625]$

$= 1265.625$

Second Iteration

Iteration	u_1	u_2	u_4	u_5
1	1031.25	1140.625	1390.625	1265.625
2	1007.813	1072.313	1320.313	1195.313
3	972.657	1035.157	1285.157	1160.157
4	955.079	1017.57	1267.57	1142.57
5	946.29	1008.79	1258.788	1133.789
6	941.894	1004.394	1254.394	1129.394

Hence $u_1 = u_7 = u_9 = u_3 = 941.894$, $u_5 = 1129.394$

$u_2 = u_8 = 1004.394$, $u_4 = u_6 = 1254.394$

HW

J. Solve $u_{xx} + u_{yy} = 0$, $0 \le x, y \le 1$ with $u(0, y) = 10 = u(1, y)$ and $u(x, 0) = 20 = u(x, 1)$ take $h = 0.25$ and apply leibmann method to 3 decimal accuracy.