

(An Autonomous Institution) DEPARTMENT OF MATHEMATICS



0

# NUMERICAL DIFFERENTIATION & INTEGRATION

# NUMERICAL DIFFERENTIATION:

It is the process of computing the value of the desivative dy for some particular value of x, from the given data  $(x_i, y_i)$ . If the values of x are equally spaced, we can use Newton's interpolation formula for equal intervals. If the values of x are unequally spaced, we can use Lagrange's interpolation formula (or) Newton's divided difference interpolation formula.

Differentiation using interpolation formulae:

Newton's forward difference formula to compute the derivatives:

Let us consider Newton's forward difference

formula,

mula,  

$$y = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!} \Delta^4 y_0 + \cdots$$

where 
$$u = \frac{\chi - \chi_0}{h}$$

i.e., 
$$y = y_0 + u \Delta y_0 + (u^2 - u) \Delta^2 y_0 + (u^3 - 3u^2 + 2u) \Delta^3 y_0$$

$$+ (u^4 - 6u^3 + 11u^2 - 6u) \Delta^4 y_0 + \cdots$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$



(An Autonomous Institution)
DEPARTMENT OF MATHEMATICS



$$\frac{dy}{dx} = \frac{1}{h} \frac{dy}{du}$$
i.e.,  $\frac{dy}{dx} = \frac{1}{h} \left\{ \Delta y_0 + \frac{(2u-1)}{2} \Delta^2 y_0 + \frac{(3u^2 - 6u + a)}{6} \Delta^2 y_0 + \frac{3u^2 - 6u + a}{6} \Delta^2 y_0 +$ 

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left\{ \Delta^2 y_0 + (u-1)\Delta^3 y_0 + (6u^2 - 18u + 11) \Delta^4 y_0 + \frac{1}{12} - \frac{1}{12} \right\}$$

$$\frac{d^{3}y}{dx^{3}} = \frac{1}{h^{3}} \left\{ \Delta^{3} y_{o} + \frac{12u - 18}{12} \Delta^{4} y_{o} + \cdots \right\} \longrightarrow \mathcal{F}$$

In particular, at  $x = x_0$ , u = 0. Hence putting u = 0 in (2), (3) & (4) we get the values of first, Second and third derivatives at  $z = x_0$ .

$$\left(\frac{dy}{dx}\right)_{\chi=\chi_{o}} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h} \left\{ \Delta y_{o} - \frac{1}{2} \Delta^{2} y_{o} + \frac{\Delta^{3} y_{o}}{3} - \frac{\Delta^{4} y_{o} + \dots y_{o}}{4} - \frac{\Delta^{4} y_{o} + \dots y_{o}}{4} \right\}$$

$$\left(\frac{d^{2}y}{dx^{2}}\right)_{\chi=\chi_{o}} = \left(\frac{d^{2}y}{dx^{2}}\right)_{u=0} = \frac{1}{h^{2}} \left\{ \Delta^{2} y_{o} - \Delta^{3} y_{o} + \frac{11}{12} \Delta^{4} y_{o} - \dots y_{o} \right\}$$

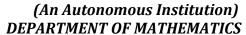
$$= \frac{1}{h} \left\{ \Delta^{2} y_{o} - \Delta^{3} y_{o} + \frac{11}{12} \Delta^{4} y_{o} - \dots y_{o} \right\}$$

$$= \frac{1}{h} \left\{ \Delta^{2} y_{o} - \Delta^{3} y_{o} + \frac{11}{12} \Delta^{4} y_{o} - \dots y_{o} \right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{\chi=\chi_0} = \left(\frac{d^3y}{dx^3}\right)_{u=0} = \frac{1}{h^3} \left\{ \Delta^3y_0 - \frac{3}{a} \Delta^4y_0 + \cdots \right\}$$

$$\longrightarrow (7)$$







Newton's Backward Difference formula to compute the derivatives:

$$\frac{dy}{dx} = \frac{1}{h} \left\{ \nabla y_n + \frac{2v+1}{2} \nabla^2 y_n + \frac{(3v^2 + 6v + 2)}{6} \nabla^3 y_n + \frac{4v^3 + 18v^2 + 22v + 6}{24} \nabla^4 y_n + \cdots \right\}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left\{ \nabla^2 y_n + (v+1) \nabla^3 y_n + \frac{6v^2 + 18v + 11}{12} \nabla^4 y_n + \dots \right\}$$

$$\frac{d^3y}{dx^2} = \frac{1}{h^3} \left\{ \nabla^3 y_n + \frac{12v + 18}{12} \nabla^4 y_n + \cdots \right\}$$

In particular, at  $x = x_n$ , v = 0. Then

$$\left(\frac{dy}{dx}\right)_{\chi=\chi_n} = \frac{1}{h} \left\{ \nabla y_n + \frac{\nabla^2 y_n}{a} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots \right\}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\chi=\chi_n} = \frac{1}{h^2} \left\{ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \cdots \right\}$$

$$\left(\frac{d^3y}{dx^3}\right)_{\chi=\chi_n} = \frac{1}{h^3} \left\{ \nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \cdots \right\}$$



# (An Autonomous Institution) DEPARTMENT OF MATHEMATICS



### Problems:

1) The population of a certain town is given below. Find the rate of growth of the population in 1931, 1941, 1961 and 1971.

Year: 1931 1941 1951 1961 1971

Population: 40.62 60.80 79.95 103.56 132.65

in thousands y

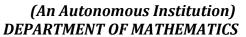
#### Solution:

(i) To get f'(1931) and f'(1941) we use forward formula.

$$U = \frac{\chi - \chi_0}{h} = \frac{1931 - 1931}{10} = 0$$

$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h} \left\{ \Delta y_0 - \frac{\Delta^2 y_0}{a} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots \right\}$$
$$= \frac{1}{10} \left\{ 20.18 - \frac{(-1.03)}{a} + \frac{(5.49)}{3} - \frac{(-4.47)}{4} \right\}$$







$$\left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{10} \left\{ 20.18 + 0.515 + 1.83 + 1.1175 \right\}$$
$$= 2.3643$$

(ii) To find 
$$y'(1941)$$
:
$$u = \frac{\chi - \chi_0}{h} = \frac{1941 - 1931}{10} = 1$$

$$\frac{dy}{dx} = \frac{1}{h} \left\{ \Delta y_0 + \frac{2u - 1}{a} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{6} \Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{24} \Delta^4 y_0 + \cdots \right\}$$

$$= \frac{1}{10} \begin{cases} 20.18 + \frac{1}{2} (-1.03) - \frac{1}{6} (5.49) + \frac{1}{4} (-4.47) \end{cases}$$

$$= \frac{1}{10} \begin{cases} 20.18 - 0.515 - 0.915 - 0.3725 \end{cases}$$

$$= 1.83775$$

(iii) To find y'(1961) and y'(1971) we use Newton's backward formula

$$V = \frac{x - x_n}{h} = \frac{1961 - 1971}{10} = -1$$

$$\frac{dy}{dx} = \frac{1}{h} \left[ \nabla y_n + \frac{2v+1}{2} \nabla^2 y_n + \frac{3v^2 + 6v + 2}{6} \nabla^3 y_n + \frac{3v^2 + 6v + 2}{6} \nabla^4 y_n + \frac{3v^2 + 2av + 6}{6} \nabla^4 y_n + \frac{3v^2 + 2$$

$$= \frac{1}{10} \left[ \frac{1}{2} \left( \frac{1}{2} \cdot \frac{1}{2} \cdot$$



(An Autonomous Institution) DEPARTMENT OF MATHEMATICS



DEPARTMENT OF MATHEMATICS

$$\frac{dy}{dx} = \frac{1}{10} \left( \frac{39.09 - 2.74 - 0.17 + 0.3725}{9} \right)$$

$$= \frac{2.6553}{k}$$
(N) To \$\frac{810d}{k} \frac{y'(1971)}{k} \cdots
$$V = \frac{x - x_n}{k} = \frac{1971 - 1971}{10} = 0$$

$$\left( \frac{dy}{dx} \right)_{x \ge 0} = \frac{1}{h} \left\{ \nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^3 y_n}{4} + \cdots \right\}$$

$$= \frac{1}{10} \left\{ 29.09 + 2.74 + 0.34 - 1.1175 \right\}$$

$$= \frac{1}{10} \left\{ 31.0525 \right\}$$

$$= \frac{3.10525}{k}$$



#### (An Autonomous Institution) **DEPARTMENT OF MATHEMATICS**



(2) A jet fighter's position on an aircraft carrier's runway was timed during landing.

1.2 1.3 1.4 1.5 t(sec): 1.0 1.1

y (m): 7.989 8.403 8.781 9.129 9.451 9.750

Where y is the distance from the end of the carrier.

Estimate velocity  $\left(\frac{dy}{dt}\right)$  and acceleration  $\left(\frac{d^2y}{dt^2}\right)$  at

(ii) t = 1.6 using numerical differentiation. (i) t = 1.1

### Solution:

Solution:  

$$\alpha$$
 y  $\Delta$ y  $\Delta^2$ y  $\Delta^3$ y  $\Delta^4$ y  $\Delta^5$ y  $\Delta^6$ y

7.989 1:0 6.414

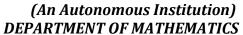
1.6 10.031

# (i) To find t=1.1:

$$u = \frac{\chi - \chi_0}{h} = \frac{1.1 - 1.0}{0.1} = 1$$

$$\left(\frac{dy}{dx}\right)_{t=1.1} = \left(\frac{dy}{dx}\right)_{u=1} = \frac{1}{h} \left[\Delta y_0 + \left(\frac{2u-1}{2}\right) \Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right) \Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right) \Delta^4 y_0 + \dots \right]$$







$$\frac{dy}{dx} = \frac{1}{0.1} \left[ 0.414 + \left( \frac{2-1}{2} \right) (-0.036) + \left( \frac{3-6-2}{6} \right) (0.006) \right] + \left( \frac{4-18+22-6}{24} \right) (-0.002) + \cdots \right]$$

$$= \frac{3.9483}{dx^2} = \frac{1}{h^2} \left[ \Delta^2 y_0 + (u-1) \Delta^3 y_0 + \left( \frac{6u^2 - 18u + 11}{12} \right) \Delta^4 y_0 + \cdots \right]$$

$$= \frac{1}{(0.1)^2} \left[ -0.036 + (1-1) (0.006) + \left( \frac{6-18+11}{12} \right) (-0.005) \right]$$

$$= -3.5833$$
(ii)  $10 \text{ kind } t = 1.6 :$ 

$$V = \frac{t-t_n}{h} = \frac{1.6-1.6}{0.1} = 0$$

$$\frac{dy}{dx} = \frac{1}{h} \left[ \nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{\nabla^4 y_n}{4} + \cdots \right]$$

$$= \frac{1}{0.1} \left[ 0.281 + \frac{1}{2} (-0.018) + \frac{1}{3} (0.005) + \frac{1}{4} (0.002) + \frac{1}{5} (0.003) + \frac{1}{6} (0.002) \right]$$

$$= \frac{2.751}{dx^2} = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \cdots \right]$$

$$= \frac{1}{(0.1)^2} \left[ -0.018 + 0.005 + \frac{11}{12} (0.002) + \cdots \right]$$

$$= -1.1167$$



# (An Autonomous Institution) DEPARTMENT OF MATHEMATICS



(5)

(3) Using the following data, find 
$$f'(5)$$
,  $f''(5)$  and the maximum value of  $f(x)$ 

$$x: 0 = 2 = 3 = 4 = 7$$
  
 $f(x): 4 = 26 = 58 = 112 = 466 = 926$ 

Solution: Since the values of x are not equally spaced, we use Newton's divided difference formula

By Newton's divided difference formula,

By Newton's divided of 
$$0$$
  $f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0)$   
 $f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^3 f(x_0) + \cdots$ 
 $+ (x - x_0)(x - x_1)(x - x_2) \Delta^3 f(x_0) + \cdots$ 

$$= 4 + (x-0) + (x-0)(x-2)(7) + (x-0)(x-2)(2-3)(1)$$

$$f(x) = x^3 + 2x^2 + 3x + 4$$

$$f'(x) = 3x^2 + 4x + 3$$

$$f''(x) = 6x + 4$$

922

9

$$f'(5) = 3(5^2) + 4(5) + 3 = 98$$

$$f''(5) = 6(5) + 4 = 34$$

$$f(x)$$
 is maximum if  $f'(x) = 0$ 

But the roots of this equation are imaginary. Hence there is no extremum value.



#### (An Autonomous Institution) **DEPARTMENT OF MATHEMATICS**



From the following table, find the value of & for which y is minimum and find this value of y.

$$x: -2 -1 0 1 2 3 4$$
  
 $y: 2 -0.25 0 -0.25 2 15.75 56$ 

solution:

Here h = 1.

For minimum value of 
$$y$$
,  $\frac{dy}{dx} = 0$ 

i.e., 
$$\frac{1}{h} \left[ \Delta y_0 + \left( \frac{2u - 1}{2} \right) \Delta^2 y_0 + \cdots \right] = 0$$
i.e., 
$$\frac{1}{1} \left[ -2 \cdot 25 + \left( \frac{2u - 1}{2} \right) (2 \cdot 5) \right] = 0$$

$$\frac{u = 1 \cdot 4}{h}$$

$$\frac{\chi - \chi_0}{h} = 1 \cdot 4 \implies \chi = 1 \cdot 4h + \chi_0 = -0 \cdot 6$$

$$y(x = -0.6) = y_0 + u \Delta y_0 + u(u-1) \Delta^2 y_0 + \dots$$

$$= 2 + (1.4)(-2.25) + (1.4)(0.4)(2.5) + (1.4)(0.4)$$

$$= -0.1476 + (1.4)(0.4)(-0.6)(-16)6 = 0$$

$$= -0.1476 + (1.4)(0.4)(-0.6)(-16)6 = 0$$