Transforms of periodic function:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace transformation of a periodic function $f(t)$ with period p given by

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Problems:

1) Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a-t, & a < t < 2a \end{cases} \quad \text{with } f(t+2a) = f(t)$$

Soln:-

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}(2a-t)\, dt \right]$$

$\int u\, dv = u v_1 - u' v_2$

$e^{-st} = v$

$u = t \qquad v_1 = \frac{-e^{-st}}{s}$

$u' = 1 \qquad v_2 = \frac{+e^{-st}}{s^2}$

$u'' = 0$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2}\right) - \left(\frac{-1}{s^2}\right)\right] + \left[\left(\frac{e^{-2as}}{s^2}\right) - \left(\frac{-ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as} - 2e^{-as}}{s^2}\right]$$

$a^2 - b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$b = e^{-as}$

$= e^{-2as}$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

② Find the Laplace transform of the half wave rectifier function, $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)\,dt\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left[1+e^{-s\pi/\omega}\right]}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$