② Find the Fourier series $f(x) = x$ in $(-\pi, \pi)$

$f(x) = x$

$f(-x) = -x = -f(x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is odd function

$\therefore a_0 = 0$ and $a_n = 0$.

The fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx \qquad [\because x \sin x \text{ is even fn}]$$

$u = x \qquad v = \sin nx$

$u' = 1 \qquad v_1 = -\frac{\cos nx}{n}$

$u'' = 0 \qquad v_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{2}{\pi}\left[-x\frac{\cos nx}{n} - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} - 0 + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n\right]$$

$$= \frac{-2}{n}(-1)^n$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx.$$

Find the Fourier series for

$$f(x) = \begin{cases} L+x & , -L < x < 0 \\ L-x & , 0 < x < L \end{cases}$$

$$\phi_1(x) = L+x$$

$$\phi_1(-x) = L-x = \phi_2(x)$$

$$\phi_2(x) = L-x$$

$$\phi_2(-x) = L+x = \phi_1(x)$$

$\therefore$ $f(x)$ is even function.

$\therefore$ The fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= \frac{2}{L}\int_0^L (L-x)\,dx$$

$$= \frac{2}{L}\left[Lx - \frac{x^2}{2}\right]_0^L = \frac{2}{L}\left[L^2 - \frac{L^2}{2}\right]$$

$$= \frac{2}{L}\left[\frac{L^2}{2}\right] = L$$

$$\boxed{a_0 = L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = L - x$ $\quad$ $v = \cos\left(\frac{n\pi x}{L}\right)$

$u' = -1$ $\quad$ $v_1 = \frac{\sin(n\pi x/L)}{(n\pi/L)}$

$u'' = 0$ $\quad$ $v_2 = \frac{-\cos\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)^2}$

$$= \frac{2}{L}\left[(L-x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}} - (-1)\frac{\left(-\cos\left(\frac{n\pi x}{L}\right)\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{\cos\left(\frac{n\pi L}{L}\right)}{\frac{n^2\pi^2}{L^2}} - 0 + \frac{\cos 0}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{2}{L}\cdot\frac{L^2}{\pi^2 n^2}\left[1 - \cos n\pi\right]$$

$$= \frac{2L}{n^2\pi^2}\left[-(-1)^n + 1\right]$$

$$\therefore f(x) = \frac{L}{2} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left[1 - (-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - (-1)^n\right)\cos\left(\frac{n\pi x}{L}\right)$$