2. Fourier Transform

Fourier Transform Pair:

The fourier transform of $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx \rightarrow (1)$$

→ complex no

Complex fourier transform of $f(x)$

$e^{isx} = \cos sx + i \sin sx$

Then the function $f(x)$ is the Inverse Fourier transform of $F(s)$ is given by,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \longrightarrow (2).$$

$e^{-isx} = \cos sx - i \sin sx$

The above eqns (1) and (2) are jointly called Fourier Transform pair.

$$f(x) = F^{-1}[F(s)] = F^{-1}[F[f(x)]]$$

1. Find the fourier transform of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

and hence deduce that $\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

Fourier Transform is

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1(\cos sx + i \overset{\text{odd}}{\sin sx})\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin sa}{s} - \frac{\sin 0}{s}\right]$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin sa}{s}\right)$$

Inverse fourier transform is $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

$$1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin sa}{s}(\cos sx - i \sin sx)\, ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^{\infty} \frac{\sin sa}{s}\cos sx\, ds - i\int_{-\infty}^{\infty} \frac{\sin sa}{s}\sin sx\, ds\right]$$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin sa}{s}\cos sx\, ds$$

$$1 = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin sa}{s}\cos sx\, ds$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin sa}{s}\cos sx\, ds$$

Put $x = 0$, $a = 1$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin s}{s}\, ds \qquad \text{put } s = t, \; ds = dt$$

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin t}{t}\, dt$$

② Find the FT of $f(x) = \begin{cases} x & \text{if } |x| \le a \\ 0 & \text{if } |x| > a \end{cases}$

$$f(x) = \begin{cases} x & \text{if } -a \le x \le a \\ 0 & \text{if } -\infty < x < -a \ \& \ a < x < \infty \end{cases}$$

$$F(s) = F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x [\cos sx + i \sin sx] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} x \cos sx \, dx + i \int_{-a}^{a} x \sin sx \, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[2i \int_{0}^{a} x \sin sx \, dx \right]$$

$$= i\sqrt{\frac{2}{\pi}} \left[x \left(\frac{-\cos sx}{s} \right) - 1 \left(\frac{-\sin sx}{s^2} \right) + 0 \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\, i \left[\left(-\frac{a \cos sa}{s} + \frac{\sin sa}{s^2} \right) - 0 \right]$$

$$= \sqrt{\frac{2}{\pi}}\, i \left[\frac{\sin sa - sa \cos sa}{s^2} \right]$$

③ Find the fourier transform of the function

$f(x) = \begin{cases} 1 - x^2 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases}$ & Hence deduce that

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cdot \cos \frac{s}{2} \, ds = \frac{3\pi}{16}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2) e^{isx} \, dx.$$

By using Bernoulie's formula,

$u = 1-x^2 \qquad v = e^{isx}$

$u' = -2x \qquad v_1 = \frac{e^{isx}}{is}$

$u'' = -2 \qquad v_2 = \frac{e^{isx}}{(is)^2} = \frac{-e^{isx}}{s^2}$

$v_3 = \frac{e^{isx}}{(is)^3} = \frac{-e^{isx}}{is^3}$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\frac{e^{isx}}{is} + 2x\left(\frac{-e^{isx}}{s^2}\right) - 2\left(\frac{-e^{isx}}{is^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[0+2\left(\frac{-e^{is}}{s^2}\right) + 2\frac{e^{is}}{is^3}\right] - \left[0+2\frac{e^{-is}}{s^2} + \frac{2e^{-is}}{is^3}\right]\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\left\{\frac{-e^{is}}{s^2} + \frac{e^{is}}{is^3} - \frac{e^{-is}}{s^2} - \frac{e^{-is}}{is^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\left(\frac{e^{is}+e^{-is}}{s^2}\right) + \frac{e^{is}}{is^3} - \frac{e^{-is}}{is^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\left[\frac{\cos s + i\sin s + \cos s - i\sin s}{s^2}\right] + \left[\frac{\cos s + i\sin s - \cos s + i\sin s}{is^3}\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-2\cos s}{s^2} + \frac{2i\sin s}{is^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$$

Inverse fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$1-x^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] [\cos sx - i \sin sx]\, ds$$

$$1-x^2 = \frac{2}{\pi}\left\{ \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx}_{\text{even}}\, ds - i \int_{-\infty}^{\infty} \underbrace{\left(\frac{\sin s - s\cos s}{s^3}\right) \sin sx}_{\text{odd}}\, ds \right\}$$

$$1-x^2 = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^8} \cos sx\, ds$$

Put $x = 1/2$.

$$1 - 1/4 = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos \frac{s}{2}\, ds$$

$$\frac{3}{4} = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos \frac{s}{2}\, ds$$

$$\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos \frac{s}{2}\, ds = \frac{3\pi}{16}.$$