Show that the fourier transform of

$$f(x)=\begin{cases} a^2-x^2 & , |x|<a \\ 0 & , |x|>a>0 \end{cases}$$ and hence find that

$$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right].$$ Hence deduce that

$$\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)dt = \frac{\pi}{4}.$$ using P.I show that

$$\int_0^\infty\left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \pi/15$$

$$f(x)=\begin{cases} a^2-x^2 & , -a<x<a \\ 0 & , -\infty<x<-a \;\&\; a<x<\infty \end{cases}$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a^2-x^2)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a^2-x^2)\cos sx\,dx+i\int_{-a}^{a}(a^2-x^2)\sin sx\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[2\int_{0}^{a}(a^2-x^2)\cos sx\,dx+i(0)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[(a^2-x^2)\frac{\sin sx}{s}-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[-2a\frac{\cos sa}{s^2}+\frac{2\sin sa}{s^3}\right] = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]$$

put $a=1$ $\quad F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$

i) Using Inverse Fourier transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3}\right] [\cos sx - i \sin sx]\, ds$$

$$= \frac{2}{\pi} \cdot 2 \int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) \cos sx\, ds$$

Put $x = 0$ $\quad f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right) ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(1-0) = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}.$$

ii) Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left\{2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3}\right)\right\}^2 ds$$

$$2\int_0^1 (1 + x^4 - 2x^2)\, dx = \frac{16}{\pi} \int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$\frac{16}{\pi} \int \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$= 2\left[1 + \frac{1}{5} - \frac{2}{3}\right]$$

$$= 2\left[\frac{15 + 3 - 10}{15}\right] = \frac{16}{15}$$

$$\int_0^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{16}{15}\left(\frac{\pi}{16}\right)$$

$$\Rightarrow \int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$$

Parseval's Identity (or) Rayleigh's Theorem

Let $F(s)$ be the Fourier transform of $f(x)$ then

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

Result:- $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

Self Reciprocal Function:

If the fourier transform of $f(x)$ is equal to $F(s)$, then $f(x)$ is said to be reciprocal function under Fourier transform

i.e) $F[f(x)] = F(s)$

eg: $F\left[e^{-x^2/2}\right] = e^{-s^2/2}$.

Show that the function $e^{-x^2/2}$ is self reciprocal under Fourier Transform.

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2-2isx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[x^2-2isx+(is)^2-(is)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{(x-is)^2}{2}\right]+\frac{(is)^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{(is)^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} dx \qquad \text{Put } t = \frac{x-is}{\sqrt{2}} \Rightarrow dt = \frac{dx}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{2}\, dt$$

$$= \frac{\sqrt{2}}{\sqrt{2\pi}} e^{-s^2/2} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} e^{-s^2/2} \cdot \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$\Downarrow$
Gaussian Integral

$$= e^{-s^2/2}.$$

$\therefore$ The fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self reciprocal under fourier transform

nd the F.T of $f(x) = \begin{cases} a - |x| & , |x| < a \\ 0 & , |x| > a > 0. \end{cases}$ and

educe that $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^2 dt$ and $\int_0^{\infty} \left(\frac{\sin t}{t}\right)^4 dt$

$$f(x) = \begin{cases} a - |x| & , -a < x < a \\ 0 & , -\infty < x < -a,\ a < x < \infty \end{cases}$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a - |x|) [\cos sx + i \sin sx] dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{a} (a - x) (\cos sx + i \sin sx) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (a - x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(a-x)\frac{\sin sx}{s} - (-1)\left(-\frac{\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a = -\sqrt{\frac{2}{\pi}}\left[\frac{\cos sa}{s^2} - \frac{\cos 0}{s^2}\right]$$

$$= -\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 sa/2}{s^2}\right]$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos\theta = 1 - 2\sin^2\theta/2$

i) using IFT

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{2\sin^2 sa/2}{s^2}\right](\cos sx - i\sin sx)\, dx$$

$$= \frac{2 \cdot 2}{\pi}\int_0^{\infty}\frac{1}{s^2}\cdot \sin^2\frac{sa}{2}\,[\cos sx]\, ds$$

Put $x=0$, $a=2$

$$f(0) = \frac{4}{\pi}\int_0^{\infty}\frac{\sin^2 s}{s^2}\, ds$$

$$\therefore \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4} f(0) = \frac{\pi}{4}(2-|0|)$$

$$= \frac{\pi}{2}.$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(ii) Parseval's Identity:-

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-2}^{2} [2-|x|]^2 dx = \int_{-\infty}^{\infty} 4 \cdot \frac{2}{\pi} \frac{\sin^4 \frac{sa}{2}}{s^4} ds$$

$$2\int_{0}^{2} (2-x)^2 dx = \frac{8}{\pi} \cdot 2 \int_{0}^{\infty} \left[\frac{\sin \frac{sa}{2}}{s}\right]^4 ds$$

$a=2$

$$2\left[\frac{(2-x)^3}{3}\right]_0^2 = \frac{16}{\pi} \int_0^\infty \left[\frac{\sin s}{s}\right]^4$$

$$\left(\frac{-2}{3}\right)(-8)\frac{\pi}{16} = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

$$\frac{\pi}{3} = \int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

Find the fourier Transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1>0 \end{cases}$ and deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ and $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$

By Fourier Transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\, dx.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s^2} + \frac{\cos 0}{s^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1-\cos s}{s^2} \right]$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$1 - \cos 2\theta = 2\sin^2\theta$

$1 - \cos\theta = 2\sin^2 \theta/2$

$$F(s) = \sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 s/2}{s^2} \right]$$

By Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} \left(\frac{2\sin^2 s/2}{s^2} \right) \right) (\cos sx - i \sin sx) \, ds$$

$$= \frac{2.2}{\pi} \int_0^{\infty} \left(\frac{\sin^2 s/2}{s^2} \right) \cos sx \, ds$$

Put $x = 0$ and $s/2 = t$ $\quad s = 2t \quad ds = 2dt$

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \left(\frac{\sin^2 t}{(2t)^2} \right) 2dt$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 t}{4t^2} \cdot 2dt \Rightarrow 1 = \frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt$$

$$\int_0^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}$$

By Parsevals Identity :

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-|x|)^2 dx = \int_{-\infty}^{\infty} \left[\sqrt{\frac{2}{\pi}} \left[\frac{2\sin^2 s/2}{s^2}\right]\right]^2 ds$$

$$2\int_{0}^{1} (1-x)^2 dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin^4 s/2}{s^4}\right) ds$$

Put $s/2 = t \quad 2t = s \quad ds = 2dt$

$$2\int_{0}^{1} (1+x^2-2x)\, dx = \frac{16}{\pi} \int_{0}^{\infty} \left(\frac{\sin^4 t}{(2t)^4}\right) 2dt$$

$$2\left[x + \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1 = \frac{16}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{16t^4} \cdot 2dt$$

$$(\cancel{1} + \tfrac{1}{3} - \cancel{1}) - (0) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin^4 t}{t^4} dt$$

$$\frac{\pi}{3} = \int_{0}^{\infty} \frac{\sin^4 t}{t^4} dt$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

Show that the fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 > 0 \end{cases}$

and hence find that $2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]$. Hence deduce

that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$. Using Parseval's Identity show

that $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

$$f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & -\infty < x < 1 \ \& \ 1 < x < \infty \end{cases}$$

By Fourier Transform,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)(\cos sx + i \sin sx) dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1 - x^2) \cos sx \, dx \quad = \sqrt{\frac{2}{\pi}} \int_0^1 (1 - x^2) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1 - x^2) \frac{\sin sx}{s} - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right] = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right]$$

By Inverse Fourier Transform,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s - s\cos s}{s^3} \right] \right] (\cos sx - i \sin sx) \, ds$$

$$= \frac{4}{\pi} \int_0^\infty \left[\frac{\sin s - s\cos s}{s^3} \right] \cos sx \, ds$$

Put $x = 0$

$$\frac{\pi}{4} f(0) = \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) ds$$

$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) ds = \frac{\pi}{4}$$

Replace s by t

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}.$$

By Parseval's Identity,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

$$\int_{-1}^{1} (1-x^2)^2 dx = \int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds$$

$$2\int_0^1 (1+x^4-2x^2)\, dx = \frac{4.2.2}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[x+\frac{x^5}{5}-\frac{2x^3}{3}\right]_0^1 = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[1+\frac{1}{5}-\frac{2}{3}\right] = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left[\frac{15+3-10}{15}\right] = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

$$2\left(\frac{8}{15}\right) = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds$$

Replace s by t

$$\frac{\pi}{15} = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 ds$$