Fourier Sine Transform:

The fourier sine transform of $f(x)$ is defined by

$$F_s[s] = F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

The inverse fourier sine transform of $F_s(s)$ is given by,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

Fourier Cosine Transform:

The fourier cosine transform of $f(x)$ is defined by

$$F_c(s) = F_c(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

The inverse fourier cosine transform of $F_c(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds$$

Parseval's Identity:

Sine transform:

If $F(s)$ is the Fourier transform of $f(x)$ then

$$\int_0^\infty [f(x)]^2\,dx = \int_0^\infty [F_s(s)]^2\,ds$$

Cosine transform:

If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_0^\infty [f(x)]^2\,dx = \int_0^\infty [F_c(s)]^2\,ds$$

1. Find the fourier sine Transform of f(x) defined as

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{-\cos sx}{s} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos s + \cos 0}{s} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{1 - \cos s}{s} \right]$$

2. Find the fourier sine Transform of $1/x$.

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx.$$

Put $\theta = sx \Rightarrow d\theta = s\,dx \Rightarrow \frac{d\theta}{s} = dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin\theta}{\theta} \, d\theta$$

$$\because \int_0^\infty \frac{\sin\theta}{\theta} \, d\theta = \frac{\pi}{2}$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

3. Find the fourier cosine transform of $2e^{-3x} + 3e^{-2x}$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[2\int_0^\infty e^{-3x} \cos sx \, dx + 3\int_0^\infty e^{-2x} \cos sx \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[2\left[\frac{3}{s^2+9} \right] + 3\left[\frac{2}{s^2+4} \right] \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{6}{s^2+9} + \frac{6}{s^2+4} \right]$$

④ Find the fourier sine & cosine transform of e^{-ax} and deduce that inverse fourier transform & parsevals identity

Sine transform:

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s}{s^2+a^2}\right]$$

Inverse Fourier sine Transform:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right) \sin sx \, ds$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds$$

$$\int_0^\infty \frac{s}{s^2+a^2} \sin sx \, ds = \frac{\pi}{2} e^{-ax}$$

Parseval's Identity:-

$$\int_0^\infty (f(x))^2 dx = \int_0^\infty (F_s(s))^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right)\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\left[\frac{e^{-\infty}}{-2a} - \frac{e^0}{-2a}\right] = \frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds$$

$$\frac{2}{\pi} \int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds = \frac{1}{2a}$$

$$\int_0^\infty \left(\frac{s}{s^2+a^2}\right)^2 ds = \frac{\pi}{4a}$$

Cosine Transform:

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right]$$

Inversion:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{s^2+a^2}\right] \cos sx \, ds$$

$$e^{-ax} = \frac{2}{\pi} \int_0^\infty \frac{a}{s^2+a^2} \cos sx \, ds$$

$$\Rightarrow \int_0^\infty \frac{a}{s^2+a^2} \cos sx \, dx = \frac{\pi}{2} e^{-ax}$$

Parseval's Identity:-

$$\int_0^\infty (f(x))^2 dx = \int_0^\infty (F(s))^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)\right)^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \left(\frac{a}{a^2+s^2}\right)^2 ds$$

$$\int_0^\infty \left(\frac{a}{s^2+a^2}\right)^2 ds = \frac{\pi}{2}\left(\frac{1}{2a}(0-1)\right)$$

$$= \frac{\pi}{4a}.$$

5. Find the Fourier Cosine Transform of $\frac{e^{-ax}}{x}$ and hence find $F_c\left[\frac{e^{-ax} - e^{-bx}}{x}\right]$

$$F_c[s] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx$$

$$\frac{d}{ds} F_c[s] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial}{\partial s}\left(\frac{e^{-ax}}{x}\cos sx\right) dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x}(-\cancel{x}\sin sx)\, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax}\sin sx\, dx$$

$$\frac{d}{ds} F_c[s] = -\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \qquad \because \int_0^\infty e^{-ax}\sin bx\, dx = \frac{b}{a^2+b^2}$$

Integrating, we get

$$F_c[s] = -\sqrt{\frac{2}{\pi}} \int \frac{s}{s^2+a^2}\, ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\int \frac{2s}{s^2+a^2}\, ds$$

$$F_c[s] = -\frac{1}{\sqrt{2\pi}} \log(s^2+a^2)$$

Similarly, $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{-1}{\sqrt{2\pi}} \log(s^2+b^2)$

Now,

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{-1}{\sqrt{2\pi}}\log(s^2+a^2) + \frac{1}{\sqrt{2\pi}}\log(s^2+b^2)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\log(s^2+b^2) - \log(s^2+a^2)\right]$$

$$= \frac{1}{\sqrt{2\pi}} \log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Properties of Fourier Transform, FCT, FST:

1. Linear Property:

Fourier Transform:

$F[af(x)+bg(x)] = aF[f(x)]+bF[g(x)]$ where a and b are real numbers.

Proof: $F[af(x)+bg(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[af(x)+bg(x)]e^{isx}dx$

$= \frac{a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx + \frac{b}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{isx}dx$

$= aF[f(x)]+bF[g(x)]$

Fourier sine Transform:

$F_S[af(x)+bg(x)] = aF_S[f(x)]+bF_S[g(x)]$

Proof:- $F_S[af(x)+bg(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty}[af(x)+bg(x)]\sin sx\,dx$

$= a\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(x)\sin sx\,dx + b\sqrt{\frac{2}{\pi}}\int_0^{\infty}g(x)\sin sx\,dx$

$= aF_S[f(x)]+bF_S[g(x)]$

Similarly, Fourier cosine Transform:

$F_C[af(x)+bg(x)] = aF_C[f(x)]+bF_C[g(x)]$

2. Change of scale Property:

For any non zero real a, $F[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right), a>0$

Proof:-

WKT, $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$

Now, $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(ax)e^{isx}dx$

Put $t=ax$

$\frac{dt}{dx}=a \qquad dx=\frac{dt}{a}$

When $x=-\infty \Rightarrow t=-\infty$

$x=\infty \Rightarrow t=\infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is\frac{t}{a}} \frac{dt}{a} \quad i\left(\frac{s}{a}\right)t$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t/a)}\, dt$$

$$= \frac{1}{a} F\left[\frac{s}{a}\right]$$

3) Shifting Property:

i) $F[f(x-a)] = e^{ias} F(s)$

ii) $F[e^{iax} f(x)] = F(s+a)$

Proof:- $F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$

i) Now, $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$

Put $t = x-a$ | $x = -\infty \Rightarrow t = -\infty$

$dt = dx$ | $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{is(t+a)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist} \cdot e^{isa} dt$$

$$= \frac{e^{isa}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$$

$$= e^{ias} F(s)$$

ii) $F[e^{iax} f(x)] = F(s+a)$

Proof: $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x} dx$

$$= F(s+a)$$

4. Modulation Property:-

Fourier Transform: If $F(s)$ is the Fourier transform of $f(x)$

then $F[f(x)\cos ax] = \frac{1}{2}[F(s+a)+F(s-a)]$

Proof:- $F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$

Now, $F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax\, e^{isx}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)e^{isx}\,dx$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\left[e^{i(s+a)x} + e^{i(s-a)x}\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s+a)x}\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i(s-a)x}\,dx\right]$$

$$= \frac{1}{2}\left[F(s+a) + F(s-a)\right]$$

Fourier Sine Transform:-

$F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

Proof:-

$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \sin sx\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx \cos ax\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,\frac{1}{2}\left[\sin(sx+ax) + \sin(sx-ax)\right]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(s-a)x\,dx\right]$$

$$= \frac{1}{2}\left[F_s(s+a) + F(s-a)\right]$$

Fourier Cosine Transform:-

$F_c[f(x)\cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

5. $F[x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$

Proof:- $F[s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$

Differentiating B.s 'n' times w.r.t x.

$$\frac{d^n}{ds^n} F(s) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{ds^n} \int_{-\infty}^{\infty} f(x) e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial s^n} [f(x) e^{isx}]\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (ix)^n e^{isx}\,dx.$$

$$= i^n \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, x^n e^{isx}\,dx$$

$$= i^n F[x^n f(x)]$$

$$\Rightarrow F[x^n f(x)] = \frac{1}{(i)^n} \frac{d^n}{ds^n} F(s)$$

$$= (-i)^n \frac{d^n}{ds^n} F(s)$$

6) i) $F[f'(x)] = -is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

ii) $F[f^n(x)] = (-i)^n s^n F(s)$ if $f(x), f'(x), \ldots f^{(n-1)}(x) \to 0$ as $x \to \pm\infty$

7) $F[\overline{f(x)}] = \overline{F(-s)}$

8) i) $F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$

ii) $F_c[x f(x)] = \frac{d}{ds} F_s[f(x)]$

1) Find the FST and FCT of $x e^{-ax}$

By property,

$$F_s[x f(x)] = -\frac{d}{ds} F_c[f(x)]$$

$$F_s[x e^{-ax}] = \frac{-d}{ds} F_c[e^{-ax}]$$

$$= \frac{-d}{ds} \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_s[x e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2+s^2)^2}$$

and $F_c[x f(x)] = \frac{d}{ds} F_s(f(x))$

$$F_c[x e^{-ax}] = \frac{d}{ds} [F_s(e^{-ax})]$$

$$= \frac{d}{ds} \left[\sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{(s^2+a^2) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

Identity Property:-

i) $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$

ii) $\int_0^\infty f(x)g(x)\,dx = \int_0^\infty F_s(s)\,G_s(s)\,ds$

(iii) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$

iv) $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$

Find the Fourier Sine Transform of $\frac{e^{-ax}}{x}$ & hence find $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$

$$F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\, dx.$$

Diff w.r.t s,

$$\frac{d}{ds} F_s\left[\frac{e^{-ax}}{x}\right] = \frac{d}{ds}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial x}(\sin sx)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\cdot \cos sx\cdot \cancel{x}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)$$

Integrating both sides by 's.'

$$\int\frac{d}{ds} F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\, a\int\frac{1}{s^2+a^2}\, ds$$

$$\int\frac{1}{s^2+a^2}\, ds = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)$$

$$= \sqrt{\frac{2}{\pi}}\cdot a\left(\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right)$$

$$= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$$

Similarly, $F_s\left(\frac{e^{-bx}}{x}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$

$$F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)\right]$$

Find Fourier sine Transform & Fourier cosine Transform of $e^{-a|x|}$. Hence show that

i) $\int_0^\infty \frac{\cos sx}{s^2+a^2}\,dx = \frac{\pi}{2a}e^{-as}$

ii) $\int_0^\infty \frac{x\sin sx}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-as}$.

By Fourier sine Transform,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

Inverse fourier sine transform of $f(x)$ is

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s[f(x)]\sin sx\,ds$$

$(x \leftrightarrow s$

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sx\,ds$$

$$= \sqrt{\frac{2}{\pi}}\cdot\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds$$

Put $s = x$ & $x = s$

$\int_0^\infty \frac{x\sin xs}{x^2+a^2}\,dx$.

$$\frac{\pi}{2}e^{-ax} = \int_0^\infty \frac{s\sin sx}{s^2+a^2}\,ds$$

Put $s = x$

$$\int_0^\infty \frac{x \sin sx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-as}$$

Fourier cosine transform,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\,dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)$$

Inverse fourier transform of

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c[f(x)] \cos sx\,ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right) \cos sx\,ds$$

$$e^{-ax} = \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{s^2+a^2}\,ds$$

$$\frac{\pi}{2a} e^{-as} = \int_0^\infty \frac{\cos sx}{s^2+a^2}\,dx.$$