Definition: Convolution:

Let $f(x)$ & $g(x)$ be the function defined in $(-\infty, \infty)$ then the convolution of $f(x)$ & $g(x)$ is defined by

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

Convolution Theorem:

The fourier transformation of convolution of $f(x)$ & $g(x)$ is the product of their fourier transformation.

$$F[f(x) * g(x)] = F(s) G(s)$$

Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

$$f(x) = e^{-ax}$$

Fourier cosine transform,

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{(s^2+a^2)}$$

Using parseval's identity property,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2}\right)\right]^2 ds$$

$$\int_0^\infty e^{-2ax}\,dx = \frac{2a^2}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds$$

$$\left(\cancel{0} - \frac{1}{-2\cancel{a}}\right)$$

$$\left(\frac{e^{-2ax}}{-2a}\right)_0^\infty = \frac{2a^2}{\pi}\int_0^\infty \left(\frac{1}{s^2+a^2}\right)^2 ds$$

$$\frac{\pi}{2a^2}\left(0 - \frac{1}{-2a}\right) = \int_0^\infty \frac{1}{(s^2+a^2)^2}\,ds$$

Put $s = x$
$ds = dx$

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx = \frac{\pi}{4a^3}.$$

Evaluate $\displaystyle\int_0^\infty \frac{x^2}{(x^2+a^2)^2}\,dx$

$f(x) = e^{-ax}$

By Fourier sine transform

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

Using Parseval's Identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\right)^2 ds$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

$$\frac{-\pi}{4a}[0-1] = \int_0^\infty \frac{s^2}{(s^2+a^2)^2} ds$$

Replace s by x

$$\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx = \frac{\pi}{4a}$$

Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$f(x) = e^{-ax}$

By Fourier cosine transform

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

$g(x) = e^{-bx}$

By FCT,

$$G_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos sx\, dx = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos sx\, dx$$

$$G_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)$$

Using Identity Property,

$$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty F_c(s)\,G_c(s)\,ds$$

$$\int_0^\infty e^{-ax}e^{-bx}\,dx = \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{s^2+a^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{s^2+b^2}\right)ds$$

$$\int_0^\infty e^{-(a+b)x}\,dx = \frac{2ab}{\pi}\int_0^\infty \frac{1}{(s^2+a^2)(s^2+b^2)}\,ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$$

$$\frac{\pi}{2ab}\left[\frac{0-1}{-(a+b)}\right] = \int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)}$$

Put $s=x$

$$\frac{\pi}{2ab(a+b)} = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}.$$

Evaluate $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$

$$f(x) = e^{-ax}$$

By FST,

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

$g(x) = e^{-bx}$

FST, $G_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \sin sx \, dx$

$$G_s(e^{-bx}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2}.$$

Using Identity property.

$$\int_0^\infty f(x) g(x) \, dx = \int_0^\infty F_s(s) G_s(s) \, ds$$

$$\int_0^\infty e^{-ax} e^{-bx} \, dx = \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+a^2}\right) \sqrt{\frac{2}{\pi}} \left(\frac{s}{s^2+b^2}\right) ds$$

$$\int_0^\infty e^{-(a+b)x} \, dx = \frac{2}{\pi} \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+a^2)} ds$$

$$\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty \frac{\pi}{2} = \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds$$

$$\frac{-\pi}{2(a+b)} \left[0-1\right] = \int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds$$

$$\frac{\pi}{2(a+b)} = \int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$$