Definition :-

The laplace transform of a function $f(t)$ defined for $0 \le t < \infty$ is $\int_0^\infty e^{-st} f(t)\, dt$ it is denoted by $L[f(t)]$.

(i.e) $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

formulas :-

$L[1] = 1/s$ $L[\sin at] = \frac{a}{s^2+a^2}$

$L[e^{at}] = \frac{1}{s-a}$ $L[\cos at] = \frac{s}{s^2+a^2}$

$L[e^{-at}] = \frac{1}{s+a}$ $L[t] = \frac{1}{s^2}$

2m **Properties of Laplace transforms :-**

i) $L[e^{at} f(t)] = [L[f(t)]]_{s \to s-a}$ [If the function is multiplies with e]

ii) $L[t f(t)] = -\frac{d}{ds} L[f(t)]$

[If the function is multiplier with t]

$L[t^2 f(t)] = \frac{d^2}{ds^2} L[f(t)]$

In general $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)]$

iii) $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\, ds$

[If the function is divisible with t]

Problems :-

1. find the laplace transforms of $e^{-3t} \cos 3t$

Soln:- $L[e^{-2t}\cos 3t]$

By the property: $L[e^{at}f(t)] = [L[f(t)]]_{s\to s-a}$

$= [L[\cos 3t]]_{s\to s+2}$

By the formula $L[\cos at] = \frac{s}{s^2+a^2}$

$\therefore [L[\cos 3t]]_{s\to s+2} = \left[\frac{s}{s^2+9}\right]_{s\to s+2} \quad (3)^2$

$= \frac{s+2}{(s+2)^2+9}$

$= \frac{s+2}{s^2+4+4s+9}$

$= \frac{s+2}{s^2+4s+13}$ //

2) find the laplace transform of $t\sin 2t$.

Soln:- $L[t\sin 2t]$

By the property: $L[tf(t)] = -\frac{d}{ds}L[f(t)]$

$= -\frac{d}{ds}L[\sin 2t]$

$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] \quad \frac{u}{v}$ $\qquad L[\sin at] = \frac{a}{s^2+a^2}$

$= -\left[\frac{(s^2+4)\cdot 0 - 2(2s)}{(s^2+4)^2}\right]$

$= \frac{+4s}{(s^2+4)^2}$ //

3) find the laplace transform of $\frac{1-e^{-t}}{t}$

Soln:- $L\left[\frac{1-e^{-t}}{t}\right]$

By the property: $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$

$$= \int_s^\infty L[1-e^{-t}]\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{1}{s+1}\right] ds$$

$$= [\log s - \log(s+1)]_s^\infty = \left[\log \frac{s}{s+1}\right]_s^\infty$$

$$= \left[\log \frac{s}{s(1+1/s)}\right]_s^\infty = \left[\log \frac{1}{1+1/s}\right]_s^\infty$$

$$= \log 1 - \log \frac{1}{1+s}$$

$$= 0 - \log\left(\frac{s}{s+1}\right)$$

$$= \log\left(\frac{s+1}{s}\right)$$

4) Find the laplace transform of $te^{-3t}\cos 2t$

Soln:- $L[te^{-3t}\cos 2t] = L[e^{-3t} + \cos 2t]$

By the 1^{st} property $\Rightarrow [L[t\cos 2t]]_{s\to s+3} \to ①$

$L[t\cos 2t] = -\frac{d}{ds} L[\cos 2t]$ [By 2^{nd} property]

$$= -\frac{d}{ds}\left[\frac{s}{s^2+4}\right]$$

$$= -\left[\frac{(s^2+4)(1 - s(2s))}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-s^2+4}{(s^2+4)^2}\right]$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

sub in (1)

$$L[te^{-3t}\cos 2t] = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{s \to s+3}$$

$$= \frac{(s+3)^2-4}{[(s+3)^2+4]^2}$$

$$= \frac{s^2+9+6s-4}{(s^2+9+6s+4)^2}$$

$$= \frac{s^2+6s+5}{(s^2+6s+13)^2} \;//$$