Transforms of periodic functions :-

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The laplace transformation of a periodic function $f(t)$ with period p given by,

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\,dt$$

1) Find the laplace transforms of the rectangular wave given by $f(t) = \begin{cases} +1, & 0<t<b \\ -1 & b<t<2b \end{cases}$

Given : $f(t) = \begin{cases} 1, & 0<t<b \\ -1, & b<t<2b \end{cases}$

Soln:- $L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$

$= \frac{1}{1-e^{-2bs}} \left[\int_0^b e^{-st} dt + \int_b^{2b} e^{-st}(-1)\, dt \right]$

$= \frac{1}{1-e^{-2bs}} \left[\frac{-1}{s} (e^{-st})_0^b + \frac{1}{s} (e^{-st})_b^{2b} \right]$

$= \frac{1}{s(1-e^{-2bs})} \left[-(e^{-bs}-1) + (e^{-2bs} - e^{-bs}) \right]$

$= \frac{-e^{-bs}+1+(e^{-bs})^2-e^{-bs}}{s(1-e^{-2bs})} = \frac{1-2e^{-bs}+(e^{-bs})^2}{s(1-e^{-2bs})}$

$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})} (1-e^{-2bs})$

$$= \frac{1}{s(1-e^{-bs})(1+e^{-bs})}(1-e^{-2bs})$$

$$= \frac{1}{s}\left(\frac{1-e^{-bs}}{1+e^{-bs}}\right)$$

$$= \frac{1}{s}\left(\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}\right)$$

$\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = \tanh\theta$

$$= 1/s \tanh\left(\frac{bs}{2}\right)$$

(2) Find the laplace transform of the half wave rectifies function $f(t) = \begin{cases} \sin\omega t, & 0<t<\pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

Soln:- $$L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + 0\right]$$

Formula

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}\cdot\omega+\omega}{s^2+\omega^2}\right]$$

Here
$x = t$
$a = -s$
$b = \omega$

$$= \frac{\omega[1+e^{-s\pi/\omega}]}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$

(3) Find the laplace transform of $f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$ with $f(t+2a) = f(t)$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} t\, dt + \int_a^{2a} e^{-st} (2a-t)\, dt \right]$$

(using Benoulli's formula)

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t \left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a + \left[-(2a-t)\frac{e^{-st}}{s} + \frac{e^{-st}}{s^2} \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) - \left(\frac{-1}{s^2} \right) \right] + \left[\left(\frac{e^{-2as}}{s^2} \right) - \left(\frac{ae^{-as}}{s} + \frac{e^{-as}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1-e^{-2as}} \left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1+e^{-2as} - 2e^{-as}}{s^2} \right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$