Convolution :-

$f(t)$, $g(t)$ are two functions defined for $t \geq 0$ than the convolution of $f(t)$ and $g(t)$ is defined by $f(t) * g(t) = (f * g)(t) = \int_0^t f(u) g(t-u)\,du$.

Convolution theorem :-

If the two laplace transforms function defined for $t \geq 0$, then $L[f(t) * g(t)] = L[f(t)] * L[g(t)] = F(s) * G(s)$

$$f(t) * g(t) = L^{-1}[F(s) * G(s)]$$

1) Using convolution theorem, find the laplace transform of

1) $\frac{s^2}{(s^2+a^2)^2}$ 2) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ 3) $\frac{s}{(s^2+a^2)(s^2+b^2)}$

4) $\frac{1}{(s+a)(s+b)}$

5) $\frac{s^2}{(s^2+a^2)^2}$

Soln:- $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$ By convolution

$$= \underset{f(t)}{\cos at} * \underset{g(t)}{\cos at}$$

$$= \int_0^t \cos\frac{au}{A}\,\cos\frac{a(t-u)}{B}\,du$$ (By definition of convolution)

$$= \int_0^t \left[\frac{1}{2}\left[\cos(au + at - au) + \cos(au - at + au)\right]\right] d$$

$\cos(A+B)$ $\cos(A-B)$

$$= 1/2 \int_0^t \left[\cos(at) + \cos(2au - at)\right] du$$

$$= 1/2 \left[u\cos at + \frac{\sin(2au - at)}{2a} \right]_0^t$$

$$= 1/2 \left[(t-0)\cos at + \frac{1}{2a}\left[\sin at + \sin at\right]\right]$$

$\sin(-\theta) = -\sin\theta$

$$= 1/2 \left[t\cos at + \frac{\sin at}{a} \right]$$

$$= \frac{1}{2a}\left[at\cos at + \sin at \right]$$

2) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Soln:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$ By convolution theorem

$f(t)$ $g(t)$

$$= \cos at * \cos at$$

$$= \int_0^t \cos\frac{au}{A} \cos b\frac{(t-a)}{B}\,du \quad (\text{By definition of convolution})$$

$$= \int_0^t \left[\frac{1}{2}\left[\cos(au+bt-bu) + \cos(au-bt+bu)\right]\right] du$$

$$\cos(A+B) \qquad \cos(A-B)$$

$$= 1/2 \int_0^t \left[\cos\left[(a-b)u+bt\right] + \cos\left[(a+b)a-bt\right]\right] du$$

$$= 1/2 \left[\frac{\sin\left[(a-b)u+bt\right]}{a-b} + \frac{\sin\left[(a+b)u-bt\right]}{a+b}\right]_0^t$$

$$= 1/2 \left[\frac{1}{a-b}\left\{\sin\left[(a-b)t+bt\right] - \sin bt\right\} + \frac{1}{a+b}\right.$$
$$\left.\left\{\sin\left[(a+b)t-bt\right] + \sin bt\right\}\right]$$

$$= 1/2 \left[\frac{1}{a-b}\left\{\sin\left[at-bt+bt\right] - \sin bt\right\} + \frac{1}{a+b}\right.$$
$$\left.\left\{\sin\left[at+bt-bt\right] - \sin bt\right\}\right]$$

$$= 1/2 \left[\frac{1}{a-b}\left\{\sin at - \sin bt\right\} + \frac{1}{a+b}\left\{\sin at + \sin bt\right\}\right]$$

$$= 1/2 \left[\sin at\left[\frac{1}{a-b} + \frac{1}{a+b}\right] + \sin bt\left[\frac{1}{a+b} - \frac{1}{a-b}\right]\right)$$

$$= 1/2 \left[\sin at\left(\frac{a+b+a-b}{a^2-b^2}\right) + \sin bt\left(\frac{a-b-a-b}{a^2-b^2}\right)\right.$$

$$= 1/2 \left[\sin at\,\frac{2a}{a^2+b^2} + \sin bt\left(\frac{-2b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right]$$