LAPLACE TRANSFORMS

Introduction:

Laplace Transformation, named after a great French Mathematician Pierre Simon De Laplace (1749-1827) who used such transformation in the "Theory of Probability".

Uses of Laplace Transformation:

1. It is used to find the solution of linear differential equations - ordinary as well as partial.

2. It helps in solving the differential equation with boundary values without finding the general solution and then finding the values of the arbitrary constants

Transformation:

A transformation is an operation which converts a mathematical expression to a different but equivalent form.

Laplace Transformation:

Let $f(t)$ be a function of t defined for $t>0$, then the Laplace transform of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$ is defined by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

Provided the integral exists.

Condition for existence of Laplace transform:

i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a, b]$ where $a > 0$.

ii) $f(t)$ should be of exponential order.

Exponential Order:-

A function $f(t)$ is said to be of exponential order if

$$\lim_{t\to\infty} e^{-st} f(t) = 0.$$

Example:-

1. t^2 is of exponential order

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} t^2$$

$$= \lim_{t\to\infty} \frac{t^2}{e^{st}} \quad \left[\frac{\infty}{\infty} \text{ Indetermined form}\right]$$

$$= \lim_{t\to\infty} \frac{2t}{se^{st}} \quad [\text{Apply L' Hospitals Rule}]$$

$$= \lim_{t\to\infty} \frac{2}{s^2 e^{st}} = \frac{2}{\infty} = 0.$$

2. e^{t^2} is not of exponential order.

$$\lim_{t\to\infty} e^{-st} f(t) = \lim_{t\to\infty} e^{-st} e^{t^2}$$

$$= \lim_{t\to\infty} e^{-st+t^2} = e^{\infty} = \infty$$

$\therefore$ e^{t^2} is not of exponential order.

Transforms of elementary functions

① $L(1) = \frac{1}{s}$ where $s > 0$.

Proof:-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L(1) = \int_0^\infty e^{-st}.1.dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}(0-1) = \frac{1}{s}$$

$$\boxed{L(1) = \frac{1}{s}}$$

② $L(K) = \frac{K}{s}$

③ $L(t) = \frac{1!}{s^2}$

$$L(t) = \int_0^\infty e^{-st}.t\,dt$$

$$= \left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$\boxed{L(t) = \frac{1!}{s^2}}$$

Bernoulli's formula.

$I = uv_1 - u'v_2 + u''v_3$

$u = t \qquad v = e^{-st}$

$u' = 1 \qquad v_1 = \frac{e^{-st}}{-s}$

$u'' = 0 \qquad v_2 = \frac{-e^{-st}}{s^2}$

④ $\boxed{L(t^2) = \frac{2!}{s^3}}$

⑤ $L(t^n) = \dfrac{\Gamma(n+1)}{s^{n+1}}$ if $s>0$ & $n>-1$.

$$L(t^n) = \int_0^\infty e^{-st}\, t^n \, dt$$

put $x = st \Rightarrow dx = s\,dt \quad \Rightarrow \frac{dx}{s} = dt$

$$L(t^n) = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}} \, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx.$$

$$\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

⑥ $L(e^{at}) = \dfrac{1}{s-a}$ if $s-a>0$.

$$L(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$\boxed{L(e^{at}) = \frac{1}{s-a} \quad \text{if } s-a>0.}$$

⑦ $L(e^{-at}) = \dfrac{1}{s+a}$ if $s+a>0$.

$$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} \, dt$$

$$= \int_0^\infty e^{-(s+a)t} \, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$L(e^{-at}) = \dfrac{1}{s+a}$ if $s+a>0$.

⑧ To find $L(\cos at)$ & $L(\sin at)$

We know that, $e^{i\theta} = \cos\theta + i\sin\theta$

$$L(e^{iat}) = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} \cdot \frac{s+ia}{s+ia}$$

$$= \frac{s+ia}{s^2+a^2}$$

$$L(\cos at + i\sin at) = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating Real & Imaginary Parts,

$$\boxed{L(\cos at) = \frac{s}{s^2+a^2} \qquad L(\sin at) = \frac{a}{s^2+a^2}.}$$

⑨ To find $L(\sinh at)$

$$L[\sinh at] = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{1}{2}L(e^{at}) - \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right] = \frac{1}{2}\left[\frac{2a}{(s-a)(s+a)}\right]$$

$$\boxed{L(\sinh at) = \frac{a}{s^2-a^2}} \quad \text{for } s^2 > a^2.$$

⑩ To find $L(\cosh at)$

$$L(\cosh at) = L\left\{\frac{1}{2}\left[e^{at} + e^{-at}\right]\right\} = \frac{1}{2}L(e^{at}) + \frac{1}{2}L(e^{-at})$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} = \frac{1}{2}\cdot\frac{2s}{s^2-a^2}$$

$$\boxed{L(\cosh at) = \frac{s}{s^2-a^2}} \quad \text{for } s^2 > a^2.$$

Problems:

① Find $L(t^8)$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(t^8) = \frac{8!}{s^{8+1}} = \frac{40320}{s^9}$$

② Find $L(t+1)^2$

$$L[(t+1)^2] = L[t^2+2t+1]$$

$$= L(t^2) + 2L(t) + L(1)$$

$$= \frac{2!}{s^3} + 2\frac{1!}{s^2} + \frac{1}{s}$$

$$= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

③ Find $L\left(\frac{1}{\sqrt{t}}\right)$

$$L\left(\frac{1}{\sqrt{t}}\right) = L(t^{-1/2})$$

$$= \frac{\Gamma(-\frac{1}{2}+1)}{s^{-1/2+1}} = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$\Gamma n+1 = n\Gamma n$ & $\Gamma 1/2 = \sqrt{\pi}$

④ $L(\sqrt{t})$

$$L(\sqrt{t}) = L(t^{1/2}) = \frac{\Gamma 1/2+1}{s^{1/2+1}} = \frac{1/2\,\Gamma 1/2}{s\sqrt{s}} = \frac{1/2\sqrt{\pi}}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

⑦ $L(e^t) \Rightarrow$

$$L(e^t) = \frac{1}{s-1}$$

⑤ $L(t^{5/2})$

$$L(t^{5/2}) = \frac{\Gamma(5/2+1)}{s^{5/2+1}} = \frac{5/2\,\Gamma(5/2)}{s^{7/2}}$$

$$= \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)}{s^{7/2}} = \frac{15/8\sqrt{\pi}}{s^{7/2}}$$

$$= \frac{15\sqrt{\pi}}{8\,s^{7/2}}.$$

⑥ $L(e^{5t})$

$$L(e^{at}) = \frac{1}{s-a}$$

$$\Rightarrow L(e^{5t}) = \frac{1}{s-5}$$

⑧ $L(e^{-7t})$

$$L(e^{-7t}) = \frac{1}{s+7}$$

⑨ $L(e^{-t}) \Rightarrow L(e^{-t}) = \frac{1}{s+1}$

⑩ Find $L(\sin 5t)$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L(\sin 5t) = \frac{5}{s^2+25}$$

⑪ Find $L(\cos 6t)$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\cos 6t) = \frac{s}{s^2+6^2} = \frac{s}{s^2+36}$$

(12) Find $L(\sin^2 2t)$

$$\sin^2 2t = \frac{1-\cos 2t}{2}$$

$$L(\sin^2 2t) = L\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2}L\{1 - \cos 2(2t)\}$$

$$= \frac{1}{2}\{L(1) - L(\cos 4t)\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$$

$$= \frac{8}{s(s^2+16)}$$

(13). Find $L(\cos^2 3t)$

$$\cos^2 3t = \frac{1+\cos 2t}{2}$$

$$L[\cos^2 3t] = L\left[\frac{1+\cos 2(3t)}{2}\right]$$

$$= \frac{1}{2}[L(1) + L(\cos 6t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

(14) Find $L(\cos^3 2t)$

$$\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$$

$$L[\cos^3 2t] = \frac{1}{4}L[\cos 3(2t) + 3\cos 2t]$$

$$= \frac{1}{4}[L(\cos 6t) + 3L(\cos 2t)]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$$

(15) Find $L(\sin^3 3t)$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$L(\sin^3 3t) = L\left[\frac{3\sin 3t - \sin 3(3t)}{4}\right] = \frac{1}{4}\left\{3L(\sin 3t) - L(\sin 9t)\right\}$$

$$= \frac{1}{4}\left[3\left(\frac{3}{s^2+9}\right) - \frac{9}{s^2+81}\right]$$

$$= \frac{1}{4}\left[\frac{9}{s^2+9} - \frac{9}{s^2+81}\right]$$

$$= \frac{9}{4}\left[\frac{1}{s^2+9} - \frac{1}{s^2+81}\right]$$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos$

(16) Find $L(\sin 2t \cos 3t)$

$$\sin A\cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$L(\sin 2t \cos 3t) = L\left[\frac{\sin(2t+3t) + \sin(2t-3t)}{2}\right]$$

$$= L\left[\frac{\sin 5t + \sin(-t)}{2}\right]$$

$$= \frac{1}{2}\left\{L(\sin 5t) - L(\sin t)\right\}$$

$$= \frac{1}{2}\left\{\frac{5}{s^2+25} - \frac{1}{s^2+1}\right\}$$

Properties :-

Change of scale property !

If $L\{f(t)\} = F(s)$, then $L[f(at)] = \frac{1}{a} F(s/a)$

Proof:-

WKt,

$$L[f(at)] = \int_0^\infty e^{-st} f(at) dt$$

put $at = x$ $\Rightarrow$ $t = x/a$

$dt = dx/a$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}.$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t) dt$$

$$= \frac{1}{a} F(s/a)$$

First Shifting property!

If $L\{f(t)\} = F(s)$ then

i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:-

i) WKt, $L[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)] dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a)$$

Second shifting Property:-

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a=u \Rightarrow dt = du$

when $t=a \Rightarrow u=0.$

$t \to \infty \Rightarrow u \to \infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} \cdot e^{-as} f(u)\,du.$$

$$= e^{-as}\int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$ Replace $u \to t$

$$L[g(t)] = e^{-as} F(s)$$

Laplace transforms of derivatives:

If $L[f(t)] = F(s)$ then $L[f'(t)] = SF(s) - f(0)$

Proof:- $L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$

Integrating by parts we get,

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)\,(-se^{-st})\,dt$$

$$= \left[e^{-\infty} f(\infty) - e^0 f(0)\right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= SF(s) - f(0)$$

Corollary:-

Let $f''(t) = S^2F(s) - Sf(0) - f'(0)$

Let $L[g'(t)] = SG(s) - g(0)$

WKT, $L[f'(t)] = SL[f(t)] - f(0)$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ & $f(0) \to f'(0)$

$$\Rightarrow L[f''(t)] = SL[f'(t)] - f'(0)$$

$$= S[S(L[f(t)] - f(0)] - f'(0)$$

$$= S^2L[f(t)] - Sf(0) - f'(0)$$

$$= S^2F(s) - Sf(0) - f'(0).$$

Laplace Transform of Integrals :

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)\,dt$ and $g(0) = 0$ then $g'(t) = f(t)$

WKT, $L[g'(t)] = sL(g(t)) - g(0)$

$= sL(g(t))$

$\Rightarrow L[g(t)] = \frac{1}{s} L[g'(t)]$ $\left\{ \because g(t) = \int_0^t f(t)\,dt \right.$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$ $g'(t) = f(t)$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Derivative of Laplace Transform (or) Laplace transform of $tf(t)$

If $L[f(t)] = F(s)$ then $L[t f(t)] = \frac{-d}{ds} F(s)$

Proof:

WKT, $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -L[t f(t)]$$

$$\Rightarrow L[tf(t)] = \frac{-d}{ds}[F(s)]$$

In General,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [F(s)]$$

Problems:

Change of scale property:-

① Find $L[\sinh 3t]$ by using change of scale property

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3} \frac{9}{s^2-9}$$

$$= \frac{3}{s^2-9}$$

② Find $L(\cos 5t)$ using change of scale property?

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{(s/5)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}.$$

(3) Given $L[f(t)] = \frac{s^2 - s+1}{(2s+1)^2(s-1)}$ applying the change of scale property show that

$$L[f(2t)] = \frac{s^2 - 2s+4}{4(s+1)^2(s-2)}$$

Soln:

$$L[f(t)] = \frac{s^2 - s + 1}{(2s+1)^2(s-1)} = F(s)$$

$$L[f(2t)] = \frac{1}{2} F(s/2)$$

$$= \frac{1}{2}\left[\frac{(s/2)^2 - (s/2)+1}{(2s/2+1)^2 (s/2-1)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2-2s+4}{4}}{(s+1)^2(s-2)/2}\right]$$

$$= \frac{1}{4}\left[\frac{s^2-2s+4}{(s+1)^2(s-2)}\right]$$

(4) Find $L[e^{5t}]$ applying change of scale property

Soln:

$$L(e^t) = \frac{1}{s-1} = F(s)$$

$$L(e^{5t}) = \frac{1}{5} F(s/5)$$

$$= \frac{1}{5} \frac{1}{(s/5 - 1)}$$

$$= \frac{1}{5} \frac{5}{s-5}$$

$$= \frac{1}{s-5}$$

First shifting theorem:

① Find $L[e^{-3t}\sin^2 t]$

Proof: $L[e^{-at} f(t)] = F(s+a)$

$$L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s\to s+3}$$

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)[(s+3)^2+4]}\right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

② Find $L(t^2 e^{-2t})$

$L[e^{-at} f(t)] = F(s+a)$

$L[e^{-2t} t^2] = [L(t^2)]_{s\to s+2}$

$$= \left[\frac{2}{s^3}\right]_{s\to s+2}$$

$$= \frac{2}{(s+2)^3}$$

(3) Find $L[e^{2t}\cos 5t]$

$$L[e^{2t}\cos 5t] = L[\cos 5t]_{s\to s-2}$$

$$= \left[\frac{s}{s^2+25}\right]_{s\to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

Second Shifting Theorem!

1. Find $L[f(t)]$ where $f(t) = \begin{cases} 0 & , 0<t<2 \\ 3 & t>2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st} f(t)dt + \int_2^\infty e^{-st} f(t)dt$$

$$= 0 + \int_2^\infty e^{-st}\cdot 3dt \qquad = 3\int_2^\infty e^{-st}dt$$

$$= \frac{-3}{s}\left[e^{-st}\right]_2^\infty \qquad = \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

2. Find the Laplace transform of

$$f(t) = \begin{cases} \sin t & , 0<t<\pi \\ 0 & , t>\pi \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt$$

$$= \int_0^\pi e^{-st}\sin t\,dt$$

$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{-e^{s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) + \frac{1}{s^2+1}$$

$$= \frac{e^{-s\pi}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

Laplace Transforms of Derivatives:

(1) Find $L[t \sin at]$

$f(t) = t \sin at$

$f'(t) = at \cos at + \sin at$

$f''(t) = a[-at \sin at + \cos at] + a \cos at$

$= 2a \cos at - a^2 t \sin at.$

$f(0) = 0, \quad f'(0) = 0.$

$L[f''(t)] = s^2 L[f(t) - s f(0) - f'(0)]$

$L[2a \cos at - a^2 t \sin at] = s^2 L[t \sin at] - s(0) - 0.$

$\Rightarrow 2a L(\cos at) - a^2 L(t \sin at) = s^2 L(t \sin at)$

$2a L(\cos at) = s^2 L(t \sin at) + a^2 L(t \sin at)$

$2a L(\cos at) = (s^2 + a^2) L(t \sin at)$

$(s^2+a^2) L(t \sin at) = 2a \frac{s}{a^2+s^2}$

$= \frac{2as}{(s^2+a^2)^2}$

(2) Find $L[t \cos at]$

Soln:- $L[t f(t)] = \frac{-d}{ds}[L(f(t))]$

$L[t \cos at] = \frac{-d}{ds}[L(\cos at)]$

$= \frac{-d}{ds}\left[\frac{s}{s^2+a^2}\right]$

$= -\left\{\frac{s^2+a^2 - s(2s)}{(s^2+a^2)^2}\right\}$

$= -\left\{\frac{s^2+a^2 - 2s^2}{(s^2+a^2)^2}\right\} = -\left\{\frac{a^2 - s^2}{(s^2+a^2)^2}\right\}$

$= \frac{s^2 - a^2}{(s^2+a^2)^2}.$

③ Find $L[te^{2t}\sin 3t]$

$$L[te^{2t}\sin 3t] = \frac{-d}{ds}\{L[e^{2t}\sin 3t]\}$$

$$= \frac{-d}{ds}\{L(\sin 3t)\}_{s\to s-2}$$

$$= \frac{-d}{ds}\left\{\left(\frac{3}{s^2+9}\right)_{s\to s-2}\right\}$$

$$= -\left\{\frac{0-3(2s)}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \left\{\frac{6s}{(s^2+9)^2}\right\}_{s\to s-2}$$

$$= \frac{6(s-2)}{[(s-2)^2+9]^2} = \frac{6(s-2)}{(s^2-4s+4+9)^2}$$

$$= \frac{6(s-2)}{(s^2-4s+13)^2}$$

④ Find $L\left[\frac{\sin 3t}{t}\right]$

Soln:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t)\,ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right)ds = \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3\cdot\frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \quad \left[\because \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \frac{\pi}{2} - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3)$$

(5) Find $L\left[t^2 e^{-2t}\cos t\right]$

$$L\left[t^2 e^{-2t}\cos t\right] = (-1)^2 \frac{d^2}{ds^2}\left\{L\left(e^{-2t}\cos t\right)\right\}$$

$$= \frac{d^2}{ds^2}\left\{L(\cos t)_{s\to s+2}\right\}$$

$$= \frac{d^2}{ds^2}\left\{\frac{s}{s^2+1}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{s^2+1-s(2s)}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \frac{d}{ds}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)^2(-2s)-(1-s^2)\,2(s^2+1)(2s)}{(s^2+1)^4}\right\}_{s\to s+2}$$

$$= \left\{\frac{(s^2+1)(-2s)-4s(1-s^2)}{(s^2+1)^3}\right\}_{s\to s+2}$$

$$= \frac{(s^2+4s+5)(-2s-4)+(4s+8)(s^2+4s+3)}{\left[(s+2)^2+1\right]^3}$$

$$= \frac{2s^3+12s^2+18s+4}{(s^2+4s+5)^3}$$

Integral of Laplace Transform (or) Laplace Transform of $\frac{f(t)}{t}$

If $L[f(t)] = F(s)$ and if $\lim_{t \to 0} \frac{f(t)}{t}$ exist then

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

Proof: $L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Integrating w.r.t. s from s to ∞, we get,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,ds\right] dt$$

$$= \int_0^\infty f(t) \left[\int_s^\infty e^{-st}\,ds\right] dt = \int_0^\infty f(t) \left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t) \left[0 - \frac{e^{-st}}{-t}\right] dt = \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

$$\therefore \quad L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds.$$

Problems:

① Find $L\left(\frac{1-\cos t}{t}\right)$

Soln: $L\left(\frac{1-\cos t}{t}\right) = \int_s^\infty L[1-\cos t]\,ds$

$$= \int_s^\infty \{L(1) - L(\cos t)\}\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log s - \log(s^2+1)^{1/2}\right]_s^\infty$$

$$= \left[\log \frac{s}{(s^2+1)^{1/2}}\right]_s^\infty \quad = \left(\log \frac{s}{s\sqrt{1+1/s^2}}\right)_s^\infty$$

$$= \left(\log \frac{1}{\sqrt{1+1/s^2}}\right)_s^\infty \quad = \log 1 - \log\left(\frac{1}{\sqrt{1+1/s^2}}\right)$$

$$= 0 - \log \frac{s}{\sqrt{s^2+1}}$$

$$= \log\left[\frac{s}{\sqrt{s^2+1}}\right]^{-1} \quad = \log\left[\frac{\sqrt{s^2+1}}{s}\right]$$

② Find $L\left(\frac{e^{-3t} - e^{-4t}}{t}\right)$

Soln: $L(e^{-3t} - e^{-4t}) = \frac{1}{s+3} - \frac{1}{s+4}$

$$L\left[\frac{e^{-3t} - e^{-4t}}{t}\right] = \int_s^\infty \left(\frac{1}{s+3} - \frac{1}{s+4}\right) ds$$

$$= \cancel{\int_s^\infty \left(\frac{1}{s+3}\right.} \quad = \left[\log(s+3) - \log(s+4)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+3}{s+4}\right)\right]_s^\infty \quad = \log\left(\frac{s+4}{s+3}\right)$$

(3) Find $L\left[\frac{1-\cos at}{t}\right]$

Soln:

$$L\left[\frac{1-\cos at}{t}\right] = \int_s^\infty L(1-\cos at)\,ds$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+a^2}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2+a^2}}\right]_s^\infty$$

$$= 0 - \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= \log\left(\frac{\sqrt{s^2+a^2}}{s}\right)$$

(4) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L[\cos at - \cos bt]\,ds$$

$$= \int_s^\infty \left[\frac{a}{a^2+s^2} - \frac{b}{s^2+b^2}\right] ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

(5) Find the laplace transform of $e^{-t}\int_0^t t\cos t\,dt$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$$

$$\left(\because L\int_0^t f(t)dt = \frac{1}{s}L[f(s)]\right)$$

$$= \left[\frac{1}{s}L(t\cos t)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds}L(\cos t)\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{s(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2} = \frac{s^2+2s+1-1}{(s+1)[s^2+2s+1+1]^2}$$

$$= \frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

⑥ Evaluate using Laplace Transform $\int_0^\infty t e^{-2t} \sin 3t \, dt$

$$\int_0^\infty t e^{-2t} \sin 3t \, dt = \left[\int_0^\infty e^{-st} (t \sin 3t) \, dt \right]_{s=2}$$

$$= \left[L(t \sin 3t) \right]_{s=2}$$

$$= \left[\frac{-d}{ds} L(\sin 3t) \right]_{s=2}$$

$$= \left[\frac{-d}{ds} \left(\frac{3}{s^2+9} \right) \right]_{s=2}$$

$$= \left[\frac{3(2s)}{(s^2+9)^2} \right]_{s=2}$$

$$= \left[\frac{6s}{(s^2+9)^2} \right]_{s=2}$$

$$= \frac{6(2)}{((2)^2+9)^2} = \frac{12}{169}$$

Initial Value Theorem:

If the Laplace Transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$ then $\underset{t \to 0}{Lt} [f(t)] = \underset{s \to \infty}{Lt} [sF(s)]$

Proof:-

WKT, $L[f'(t)] = sL[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$$\Rightarrow sF(s) = L[f'(t)] + f(0)$$

$$sF(s) = \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

Taking limit as $s \to \infty$ on both sides we get,

$$\underset{s \to \infty}{Lt}\, sF(s) = \underset{s \to \infty}{Lt} \left\{ \int_0^\infty e^{-st} f'(t)\,dt + f(0) \right\}$$

$$= \underset{s \to \infty}{Lt} \int_0^\infty e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty \underset{s \to \infty}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= 0 + f(0)$$

$$= \underset{t \to 0}{Lt}\, f(t)$$

Hence $\underset{t \to 0}{Lt}\, f(t) = \underset{s \to \infty}{Lt}\, sF(s)$

Final Value Theorem:

If the Laplace Transform of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$ then $\underset{t\to\infty}{Lt}\,[f(t)] = \underset{s\to 0}{Lt}\,[sF(s)]$

Proof:

WKT, $L[f'(t)] = sL[f(t)] - f(0)$

$= sF(s) - f(0)$

$\Rightarrow sF(s) = L[f'(t)] + f(0)$

Taking limit $s\to 0$ on both sides, we get

$$\underset{s\to 0}{Lt}\,[sF(s)] = \underset{s\to 0}{Lt}\left\{\int_0^\infty e^{-st} f'(t)\,dt + f(0)\right\}$$

$$= \int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)\,dt + f(0)$$

$$= \int_0^\infty f'(t)\,dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0) = \underset{t\to\infty}{Lt}\, f(t)$$

Hence $\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\,[sF(s)]$

① Verify the initial and final value theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Soln: $F(s) = L[1 + e^{-t}\sin t + e^{-t}\cos t]$

$= L(1) + L(\sin t)_{s\to s+1} + L(\cos t)_{s\to s+1}$

$= \frac{1}{s} + \left(\frac{1}{s^2+1}\right)_{s\to s+1} + \left(\frac{s}{s^2+1}\right)_{s\to s+1}$

$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$

$$= \frac{1}{s} + \frac{s+2}{s^2+2s+2}$$

$$s\,F(s) = s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

Initial Value theorem:- $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+1 = 2$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2+2s}{s^2+2s\left(1+\frac{2}{s^2+2s}\right)}\right]$$

$$= \lim_{s\to\infty} 1 + \frac{s^2\left[1+2/s\right]}{s^2\left[1+\frac{2}{s}\left(1+\frac{2}{s^2+2s}\right)\right]}$$

$$= \lim_{s\to\infty} 1 + \frac{\left[1+2/s\right]}{1+2/s\left(1+2/s^2+2s\right)}$$

$$= 1+1$$

$$= 2$$

Hence $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s) = 2$

$\therefore$ Initial value theorem is verified

Final value theorem: $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

$$\underset{t\to\infty}{Lt} f(t) = \underset{t\to\infty}{Lt} \left[1+e^{-t}(\sin t + \cos t)\right] = 1$$

$$\underset{s\to 0}{Lt} sF(s) = \underset{s\to 0}{Lt} s\left[\frac{1}{s} + \frac{s+2}{s^2+2s+2}\right]$$

$$= \underset{s\to 0}{Lt} \left[1 + \frac{s^2+2s}{s^2+2s+2}\right] = 1$$

Hence $\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} sF(s) = 1$, Hence Final value theorem is verified.

Laplace Transform of Some Special functions :

Unit Step function:-

The unit step function also called Heaviside unit function is defined as,

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

This is the unit step functions at $t=a$. It can also be denoted by $H(t-a)$ or $u_a(t)$

Result:- Laplace Transform of unit step function is $\frac{e^{-as}}{s}$

(i.e) $L[u(t-a)] = \frac{e^{-as}}{s}$

Proof:-

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\,dt$$

$$= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s} \quad (s>0)$$

Transforms of periodic function:

A function $f(x)$ is said to be periodic if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace transformation of a periodic function $f(t)$ with period p given by

$$L[f(t)] = \frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Problems:

1) Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases} \quad \text{with } f(t+2a) = f(t)$$

Soln:- $$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st}\, t\, dt + \int_a^{2a} e^{-st}(2a-t)\, dt \right]$$

$\int u\, dv = uv_1 - u'v_2$

$e^{-st} = v$

$u = t \quad v_1 = \frac{-e^{-st}}{s}$

$u' = 1 \quad v_2 = \frac{+e^{-st}}{s^2}$

$u'' = 0$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{s^2}\right) \right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right]_a^{2a} \right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[\left(-a\frac{e^{-as}}{s} - \frac{e^{-as}}{s^2}\right) - \left(\frac{-1}{s^2}\right)\right] + \left[\left(\frac{e^{-2as}}{s^2}\right) - \left(\frac{-ae^{-as}}{s} + \frac{e^{-as}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$(a-b)^2 = a^2 - 2ab + b^2$

$b = e^{-as}$

$= e^{-2as}$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as} - 2e^{-as}}{s^2}\right]$$

$a^2 - b^2$

$$= \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

(2) Find the Laplace transform of the half wave rectifier function, $f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}(0)\,dt.\right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-s\pi/\omega}\,\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left[1+e^{-s\pi/\omega}\right]}{(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega})(s^2+\omega^2)}$$

$$= \frac{\omega}{(1-e^{-s\pi/\omega})(s^2+\omega^2)}$$

Inverse Laplace Transform

If the Laplace Transform of $f(t)$ is $F(s)$ ie) $L[f(t)] = F(s)$ then $f(t)$ is called an inverse Laplace transform of $F(s)$ and is written as $f(t) = L^{-1}[F(s)]$ where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace Transform

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
① $L(1) = 1/s$	$L^{-1}(1/s) = 1$
② $L(t) = 1/s^2$	$L^{-1}(1/s^2) = t$
③ $L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
④ $L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
⑤ $L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
⑥ $L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
⑦ $L\left(\frac{\sin at}{a}\right) = \frac{1}{s^2+a^2}$	$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$
⑧ $L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

④ Change of Scale Property

$$L^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$$

⑤ Multiplication by S!

If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$

then $L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$

Note!

If $f(0) \neq 0$, then $L^{-1}[sF(s)] = \frac{d}{ds} L^{-1}[F(s)] + f(0)\delta(t)$

Problem Identification:-

If $L^{-1}\left[\frac{s}{\text{quadratic eqn}}\right]$ then use result 5

If $L^{-1}\left[\frac{s}{\text{Linear eqn}}\right]$ then use the above note

⑥ Division by S:

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

⑦ Inverse Laplace transform of derivatives:

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$

Problem Identification:-

If $L^{-1}\left[\frac{s + \text{any term}}{(\text{quadratic eqn})^2}\right]$ then we use the above result.

⑧ Note:-

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$

Problem Identification:

If L^{-1}[log function or cot fn or tan fn] then we use the above result.

Problems:-

1. Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln:-

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s^2+a^2-a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+b^2)} - \frac{a^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+b^2}\right] - a^2 L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b} L^{-1}\left[\frac{b}{s^2+b^2}\right] - \frac{a^2}{b^2-a^2} L^{-1}\left(\frac{b^2-a^2}{(s^2+a^2)(s^2+b^2)}\right)$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2} L^{-1}\left[\frac{1}{s^2+a^2} - \frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\sin bt - \frac{a^2}{b^2-a^2}\left[\frac{1}{a}\sin at - \frac{1}{b}\sin bt\right]$$

② Find $L^{-1}\left[\frac{2s-5}{9s^2-25}\right]$

$$L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$$

$$= L^{-1}\left[\frac{2s}{9\left(s^2-\frac{25}{9}\right)} - \frac{5}{9\left(s^2-25/9\right)}\right]$$

$$= L^{-1}\left[\frac{2s}{9\left(s^2-\left(\frac{5}{3}\right)^2\right)} - \frac{5}{9\left(s^2-\left(5/3\right)^2\right)}\right]$$

$$= \frac{2}{9} L^{-1}\left[\frac{s}{s^2-(5/3)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{5/3}{s^2-(5/3)^2}\right]$$

$$= \frac{2}{9} \cosh\left(\frac{5}{3}\right)t - \frac{1}{3} \sinh\left(\frac{5}{3}\right)t$$

(3) Find $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = \frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s+2)^2+4}\right)\right]$$

$$= \frac{d}{dt}\left[e^{-2t} L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{2} L^{-1}\left(\frac{2}{s^2+2^2}\right)\right]$$

$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2} \sin 2t\right)$$

$$= \frac{1}{2} \frac{d}{dt}\left(e^{-2t}\sin 2t\right)$$

$$= \frac{1}{2}\left[e^{-2t}\cos 2t(2) + \sin 2t(-2)e^{-2t}\right]$$

$$= e^{-2t}\cos 2t - e^{-2t}\sin 2t$$

$$= e^{-2t}\left[\cos 2t - \sin 2t\right]$$

(4) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

<u>soln</u>:- $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log\frac{s+1}{s-1}\right)\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\log(s+1) - \log(s-1)\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{-1}{t}\left(e^{-t} - e^{t}\right) = \frac{e^{t}-e^{-t}}{t}$$

$$= \frac{2}{t}\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$= \frac{2}{t}\sinh t$$

Partial Fractions:

1. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Soln:

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \rightarrow (1)$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

Put $s=-1$

$$1 = B(-1)(-1+2)$$

$$-B = 1 \qquad \boxed{B=-1}$$

Put $s=-2$

$$1 = C(-2)(-2+1)$$

$$1 = 2C \qquad \boxed{C = 1/2}$$

Put $s=0$

$$1 = A(0+2)(0+1) \Rightarrow 2A = 1$$

$$\boxed{A = 1/2}$$

Sub in (1)

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} + \frac{(-1)}{s+1} + \frac{1}{2(s+2)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}$$

$$= \frac{1}{2}\left[1 - 2e^{-t} + e^{-2t}\right]$$

(2) Find $L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right]$

Soln :-

$$\frac{s^2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$s^2 = A(s^2+4) + (Bs+C)(s+1)$$

Put $s = -1$

$$(-1)^2 = A((-1)^2 + 4) + (B(-1)+C)(-1+1)$$

$$1 = A(5)$$

$$\boxed{A = 1/5}$$

Put $s = 0$

$$0 = A(0+4) + (B(0)+C)(0+1)$$

$$0 = 4A + C$$

$$C = -4A$$

$$\boxed{C = -4/5}$$

Put $s = -4$

$$16 = A(16+4) + (B(-4)+C)(-4+1)$$

$$16 = 20A + (-4B+C)(-3)$$

$$16 = 20(1/5) + 12B - 3(-4/5)$$

$$16 = 4 + 12B + 12/5$$

$$12B = 16 - 4 - 12/5$$

$$12B = 12 - 12/5$$

$$B = 1 - 1/5$$

$$\boxed{B = 4/5}$$

$$L^{-1}\left[\frac{s^2}{(s+1)(s^2+4)}\right] = L^{-1}\left[\frac{1/5}{s+1} + \frac{(4/5)(s - 4/5)}{s^2+4}\right]$$

$$= \frac{1}{5} L^{-1}\left[\frac{1}{s+1}\right] + \frac{4}{5} L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{4}{5}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{1}{5} e^{-t} + \frac{4}{5} \cos 2t - \frac{4}{5} \frac{\sin 2t}{2}$$

$$= \frac{1}{5} e^{-t} + \frac{4}{5} \cos 2t - \frac{4}{10} \sin 2t$$

(3) Find $L^{-1}\left[\frac{s^2+2s+1}{(s+3)(s-3)(s+1)}\right]$ ANS: $\frac{1}{3} e^{-3t} + \frac{2}{3} e^{3t}$

(4) Find $L^{-1}\left[\frac{s}{(s-3)(s^2+4)}\right]$ ANS: $\frac{3}{13} e^{3t} - \frac{3}{13} \cos 2t + \frac{2}{13} \sin 2t$

Convolution :-

If $f(t)$ and $g(t)$ are two functions defined for $t \geq 0$ then the convolution of $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(u)\, g(t-u)\, du$$

Note :- $f(t) * g(t) = g(t) * f(t)$

Convolution Theorem :-

If $f(t)$ and $g(t)$ are two Laplace transformable functions defined for $t \geq 0$ then $L[f(t) * g(t)]$ is given by,

$$L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$$

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Problems :-

① Using convolution theorem, find the inverse transform of

i) $\dfrac{s^2}{(s^2+a^2)^2}$ ii) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ iii) $\dfrac{1}{(s+a)(s+b)}$

Soln :-

i) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{s}{(s^2+a^2)} \cdot \dfrac{s}{(s^2+a^2)}\right]$

$= L^{-1}\left[\dfrac{s}{s^2+a^2}\right] * L^{-1}\left[\dfrac{s}{s^2+a^2}\right]$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \, \cos a(t-u)\, du$$

$$= \int_0^t \left[\frac{\cos(au+at-au)+\cos(au-at+au)}{2}\right] du$$

$\cos A \cos B = \dfrac{\cos(A+B)+\cos(A-B)}{2}$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au-at)]\, du$$

$$= \frac{1}{2}\left[\cos at \cdot u + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[\cos at \cdot t + \frac{\sin(2at-at)}{2a} - \frac{\sin(0-at)}{2a}\right]$$

$\sin(-\theta) = -\sin\theta$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{2\sin at}{2a}\right] = \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}[at\cos at + \sin at]$$

(ii) $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right]$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{b}\cdot\frac{b}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * \frac{1}{b}L^{-1}\left[\frac{b}{s^2+b^2}\right] = \cos at * \frac{1}{b}\sin bt$$

$$= \frac{1}{b}\int_0^t \cos au \, \sin b(t-u)\, du.$$

$\sin(A+B)+\sin(A-B) =$

$2\cos A \sin B$

$$= \frac{1}{b}\int_0^t \frac{\sin(bt-bu+au) + \sin(bt-bu-au)}{2}\,du.$$

$$\because \sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$

$$= \frac{1}{2b}\int_0^t \sin(bt+(a-b)u) + \sin[bt-(a+b)u]\,du$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b} - \frac{\cos(bt-(a+b)u)}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2b}\left[\frac{\cos(bt-(a+b)u)}{a+b} - \frac{\cos(bt+(a-b)u)}{a-b}\right]_0^t$$

$$= \frac{1}{2b}\left\{\left[\frac{\cos(\cancel{bt}-at-\cancel{bt})}{a+b} - \frac{\cos(\cancel{bt}+at-\cancel{bt})}{a-b}\right]\right.$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\left. - \left[\frac{\cos bt}{a+b} - \frac{\cos bt}{a-b}\right]\right\}$$

$$= \frac{1}{2b}\left\{+\frac{\cos at}{a+b} - \frac{\cos at}{a-b} - \frac{\cos bt}{a+b} + \frac{\cos bt}{a-b}\right\}$$

$$= \frac{1}{2b}\left\{\frac{1}{a+b}[\cos at - \cos bt] - \frac{1}{a-b}[\cos at - \cos bt]\right\}$$

$$= \frac{\cos at - \cos bt}{2b(a^2-b^2)}\left\{\frac{\cancel{a}+b-\cancel{a}-b}{a^2-b^2}\right\} = \frac{(\cos at - \cos bt)(-2b)}{2b(a^2-b^2)}$$

$$= \frac{\cos bt - \cos at}{a^2-b^2}.$$

(iii) $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{s+a} \cdot \frac{1}{s+b}\right]$$

$$= L^{-1}\left[\frac{1}{s+a}\right] \cdot L^{-1}\left[\frac{1}{s+b}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} \cdot e^{-b(t-u)} \, du$$

$$= \int_0^t e^{-au-bt+bu} \, du$$

$$= \int_0^t e^{-bt-(a-b)u} \, du.$$

$$= \left[\frac{e^{-bt-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \left[\frac{e^{-bt-at+bt}}{-(a-b)} - \frac{e^{-bt}}{-(a-b)}\right]$$

$$= -\left[\frac{e^{-at}}{a-b} + \frac{e^{-bt}}{a-b}\right]$$

$$= -\left[\frac{e^{-at} + e^{-bt}}{a-b}\right]$$

Application of Laplace transforms to Differential equations:-

If $L[f(t)] = F(s)$ then

$L[y'(t)] = sL(y) - y(0)$

$L[y''(t)] = s^2L(y) - sy(0) - y'(0)$

① Solve the differential equations using Laplace Transform $y''+4y'+4y = e^{-t}$ given that $y(0)=0$, and $y'(0)=0$.

$$y''+4y'+4y = e^{-t}$$

Taking Laplace Transform on both sides,

$$L(y''+4y'+4y) = L(e^{-t})$$

$$L(y'') + 4L(y') + 4L(y) = \frac{1}{s+1}$$

$$[s^2L(y) - sy(0) - y'(0)] + 4[sL(y) - y(0)] + 4L(y) = \frac{1}{s+1}$$

Given: $y(0)=0$, $y'(0)=0$.

$$[s^2L(y) - s(0) - 0] + 4[sL(y) - (0)] + 4L(y) = \frac{1}{s+1}$$

$$s^2L(y) + 4sL(y) + 4L(y) = \frac{1}{s+1}$$

$$(s^2+4s+4)L(y) = \frac{1}{s+1}$$

$$L(y)(s+2)^2 = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)(s+2)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$

Put $s=-2 \Rightarrow 1 = A(0) + B(0) + C(-2+1)$

$$\boxed{C=-1}$$

Put $s=-1 \Rightarrow 1 = A(-1+2)^2$

$$\boxed{A=1}$$

Put $s=0 \Rightarrow 1 = A(2)^2 + B(2)(1) + C(1)$

$$1 = 4A + 2B + C$$

$$1 = 4(1) + 2B + (-1)$$

$$1 = 2B + 3$$

$$2B = -2 \qquad \boxed{B=-1}$$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$y = L^{-1}\left(\frac{1}{(s+1)(s+2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) - L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$= e^{-t} - e^{-2t} - te^{-2t}$$