13M Applications of Laplace transforms two differential equations:-

Let $L[f(t)] = F(s)$

Then $L[y'(t)] = s\,L[y(t)] - y(0)$

$L[y''(t)] = s^2 L[y(t)] - s\,y(0) - y'(0)$

(1) Solve the differential equations using laplace transform, $y'' + 4y' + 4y = e^{-t}$ given $y(0)=0$, $y'(0)=0$

Soln:- $y'' + 4y' + 4y = e^{-t}$

$y(0) = 0,\ y'(0) = 0$

$L[y''] + 4L[y'] + 4L[y] = L[e^{-t}]$

$s^2 L[y(t)] - s\,y(0) - y'(0) + 4[sL[y(t)] - y(0)] + 4L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] - 0 - 0 + 4[sL[y(t)] - 0] + 4L[y(t)] = \frac{1}{s+1}$

$s^2 L[y(t)] + 4s[y(t)] + 4L[y(t)] = \frac{1}{s+1}$

$[s^2 + 4s + 4]\,L[y(t)] = \frac{1}{s+1}$

$L[y(t)] = \frac{1}{(s+1)(s^2+4s+4)} = \frac{1}{(s+1)(s+2)(s+2)}$

$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] \rightarrow (1)$

By PFM,

$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \longrightarrow (2)$

$= \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}$

$1 = A(s+2)^2 + B(s+1) + C(s+1)$

Put $s=-1$,

$1 = A(-1+2)^2 + 0 + 0$

$\boxed{A=1}$

Put $s=-2$,

$1 = 0 + 0 + C(-2+1)$

$1 = -C$

$\boxed{C=-1}$

Put $s=0$, $1 = 4A + 2B + C$

$4(1) + 2B - 1 = 1$

$2B = 1-3 = -2$

$\boxed{B=-1}$

(2) eq'n substitute

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

(1) $\rightarrow$ $y(t) = e^{-t} - e^{-2t} - te^{-2t}$

2) Solve $y'' - 3y' + 2y = e^{3t}$ given $y(0)=1$, $y'(0)=0$

Soln:-

$y'' - 3y' + 2y = e^{3t}$

$y(0)=1$, $y'(0)=0$

$$s^2 L[y(t)] - s\,y(0) - y'(0) - 3[sL\,y(t)] - 1] + 2L[y(t)] = \frac{1}{s-3}$$

$$s^2 L[y(t)] - s - 0 - 3[sL\,y(t)] - 1] + 2L[y(t)] = \frac{1}{s-3}$$

$$s^2 L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s-3}$$

$$(s^2 - 3s + 2) L[y(t)] = \frac{1}{s-3} + s - 3$$

$$(s-1)(s-2) L[y(t)] = \frac{1 + (s-3)(s-3)}{s-3}$$

$$= \frac{1 + s^2 + 9 - 6s}{s-3}$$

$$(s-1)(s-2) L[y(t)] = \frac{s^2 - 6s + 10}{s-3}$$

$$L[y(t)] = \frac{s^2 - 6s + 10}{(s-3)(s-1)(s-2)}$$

$$y(t) = L^{-1}\left[\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}\right] \longrightarrow (1)$$

By PFM,

$$\frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$s^2 - 6s + 10 = A(s-3)(s-2) + B(s-1)(s-3) + C(s-1)(s-2)$$

put $s = 1$,

$$1 - 6 + 10 = A(-1)(-2) + 0 + 0$$

$$5 = 2A$$

$$\boxed{A = 5/2}$$

Put $s = 2$

$$4 - 12 + 10 = A(-1)(0) + B(1)(-1) + C(1)(0)$$

$$-8 + 10 = 0 + B(-1) + 0$$

$$2 = -B$$

$$\boxed{B = -2}$$

Put $s = 3$

$$9 - 18 + 10 = 0 + 0 + C(2)(1)$$

$1 = 2C$

$\Rightarrow \boxed{C = 1/2}$

$(2) \Rightarrow \dfrac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \dfrac{5/2}{s-1} - \dfrac{2}{s-2} + \dfrac{1/2}{s-3}$

$L^{-1}\left[\dfrac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)}\right] = 5/2\, L^{-1}\left[\dfrac{1}{s-1}\right] - 2\, L^{-1}\left[\dfrac{1}{s-2}\right] + 1/2\, L^{-1}\left[\dfrac{1}{s-3}\right]$

$(1) \Rightarrow$

$y(t) = 5/2\, e^{t} - 2e^{2t} + 1/2\, e^{3t}$

Solve $y'' - 3y' + 2y = e^{-t}$ given $y(0) = 1$, $y'(0) = 0$

Soln:-

$y'' - 3y' + 2y = e^{-t}$

$y(0) = 1$, $y'(0) = 0$

$\rightarrow L[y''] - 3L[y'] + 2L[y] = L[e^{-t}]$

$s^2 L[y(t)] - s\,y(0) - y'(0) - 3[s\,L\,y(t)] - y(0)] + 2L[y(t)] = \dfrac{1}{s+1}$

$s^2 L[y(t)] - s(1) - 0 - 3[s\,L\,y(t)] - 1] + 2L[y(t)] = 1/s+1$

$s^2 L[y(t)] - s - 3s\,L[y(t)] + 3 + 2L[y(t)] = \dfrac{1}{s+1}$

$$(s^2 - 3s + 2)\, L[y(t)] = \frac{1}{s+1} + s - 3$$

$$(s^2 - 3s + 2)\, L[y(t)] = \frac{1 + (s+1)(s-3)}{s+1}$$

$$[s^2 - 3s + 2]\, L[y(t)] = \frac{1 + s^2 - 3s + s - 3}{s+1}$$

$$[s^2 - 3s + 2]\, L[y(t)] = \frac{s^2 - 2s - 2}{s+1}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s+1)[s^2-3s+2]}$$

$$L[y(t)] = \frac{s^2-2s-2}{(s+1)(s-2)(s-1)}$$

By PFM,

$$\frac{s^2-2s-2}{(s+1)(s-2)(s-1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-1}$$

$$\frac{s^2-2s-2}{(s+1)(s-2)(s-1)} = \frac{A(s-2)(s-1) + B(s+1)(s-1) + C(s+1)(s-2)}{(s+1)(s-2)(s-1)}$$

$$s^2-2s-2 = A(s-2)(s-1) + B(s+1)(s-1) + C(s+1)(s-2)$$

Put $s=-1$

$$(-1)^2-2(-1)-2 = A(-3)(-2) + B(0)(-2) + C(0)(-3)$$

$$1+2-2 = A(6)$$

$$1 = A(6)$$

$$\boxed{A = 1/6}$$

Put $s=2$,

$$(2)^2-2(2)-2 = A(s-2)(s-1) + B(s+1)(s-1) + C(s+1)(s-2)$$

$$4-4-2 = 0 + B(3)$$

$$-2 = B(3)$$

$$B = -2/3$$

Put $s = 1$,

$$(1)^2 - 2(1) - 2 = A(1-2)(1-1) + B(0) + C(1+1)(1-2)$$

$$1 - 2 - 2 = A(0) + B(0) + C(2)(-1)$$

$$1-4 = C(-2)$$

$$-3 = C(-2)$$

$$C = 3/2$$

$$\Rightarrow \left[\frac{s^2-2s-2}{(s+1)(s-2)(s-1)}\right] = \frac{\frac{1}{6}}{s+1} + \frac{\frac{-2}{3}}{s-3} + \frac{3/2}{s-1}$$

$$L^{-1}\left[\frac{s^2-2s-2}{(s+1)(s-2)(s-1)}\right] = \frac{1}{6} L^{-1}\left[\frac{1}{s+1}\right] - \frac{2}{3}\left[\frac{1}{s-3}\right] + 3/2\left[\frac{1}{s-1}\right]$$

$$y(t) \Rightarrow 1/6\, e^{-t} - 2/3\, e^{3t} + 3/2\, e^{-t}$$