convalution:-

$f(t)$, $g(t)$ are two functions defined for $t \geq 0$ than the convolution of $f(t)$ and $g(t)$ is defined by $f(t) * g(t) = (f * g)(t) = \int_0^t f(u) g(t-u)\,du$.

Convolution theorem:-

If the two laplace transforms function defined for $t \geq 0$, then $L[f(t) * g(t)] = L[f(t)] * L[g(t)] = F(s) * G(s)$

$$f(t) * g(t) = L^{-1}[F(s) * G(s)]$$

1) Using convolution theorem, find the laplace transform of

i) $\frac{s^2}{(s^2+a^2)^2}$ 2) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ 3) $\frac{s}{(s^2+a^2)(s^2+b^2)}$

4) $\frac{1}{(s+a)(s+b)}$

i) $\frac{s^2}{(s^2+a^2)^2}$

Soln:- $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$

$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+a^2}\right]$ By convolution

$$= \underset{f(t)}{\cos at} * \underset{g(t)}{\cos at}$$

$$= \int_0^t \underbrace{\cos au}_{A} \underbrace{\cos a(t-u)\,du}_{B} \quad \text{(By definition of convolution)}$$

$$= \int_0^t \left[\frac{1}{2} \left[\underset{\cos(A+B)}{\cos(au + at - au)} + \underset{\cos(A-B)}{\cos(au - at + au)} \right] du \right]$$

$$= 1/2 \int_0^t \left[\cos(at) + \cos(2au - at) \right] du$$

$$= 1/2 \left[u \cos at + \frac{\sin(2au - at)}{2a} \right]_0^t$$

$$= 1/2 \left[(t-0) \cos at + 1/2a \left[\sin at + \sin at \right] \right]$$

$\sin(-\theta) = -\sin\theta$

$$= 1/2 \left[t \cos at + \frac{\sin at}{a} \right]$$

$$= \frac{1}{2a} \left[at \cos at + \sin at \right]$$

2) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

Soln:- $L^{-1} \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)} \right] = L^{-1} \left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2} \right]$

$$= L^{-1} \left[\frac{s}{s^2+a^2} \right] * L^{-1} \left[\frac{s}{s^2+b^2} \right] \quad \text{By convolution theorem}$$

$$= \underset{f(t)}{\cos at} * \underset{g(t)}{\cos at}$$

$= \int_0^t \underbrace{\cos au}_{A} \; \underbrace{\cos b(t-u)}_{B}\, du$ (By definition of convolution)

$= \int_0^t \left[\frac{1}{2}\left[\underbrace{\cos(au + bt - bu)}_{\cos(A+B)} + \underbrace{\cos(au - bt + bu)}_{\cos(A-B)}\right]\right] du$

$= 1/2 \int_0^t \left[\cos\left[(a-b)u + bt\right] + \cos\left[(a+b)a - bt\right]\right] du$

$= 1/2 \left[\frac{\sin\left[(a-b)u + bt\right]}{a-b} + \frac{\sin\left[(a+b)u - bt\right]}{a+b}\right]_0^t$

$= 1/2 \left[\frac{1}{a-b}\left\{\sin\left[(a-b)t + bt\right] - \sin bt\right\} + \frac{1}{a+b}\right.$
$\left.\left\{\sin\left[(a+b)t - bt\right] + \sin bt\right\}\right]$

$= 1/2 \left[\frac{1}{a-b}\left\{\sin\left[at - bt + bt\right] - \sin bt\right\} + \frac{1}{a+b}\right.$
$\left.\left\{\sin\left[at + bt - bt\right] - \sin bt\right\}\right]$

$= 1/2 \left[\frac{1}{a-b}\left\{\sin at - \sin bt\right\} + \frac{1}{a+b}\left\{\sin at + \sin bt\right\}\right]$

$= 1/2 \left[\sin at\left[\frac{1}{a-b} + \frac{1}{a+b}\right] + \sin bt\left[\frac{1}{a+b} - \frac{1}{a-b}\right]\right)$

$= 1/2 \left[\sin at\left(\frac{a+b+a-b}{a^2-b^2}\right) + \sin bt\left(\frac{a-b-a-b}{a^2-b^2}\right)\right)$

$= 1/2 \left[\sin at \frac{2a}{a^2+b^2} + \sin bt\left(\frac{-2b}{a^2-b^2}\right)\right]$

$= \frac{1}{a^2-b^2}\left[a\sin at - b\sin bt\right]$