



J. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ using transforms.

(or)

Find the Fourier cosine transform of $f(x) = e^{-ax}$

and $g(x) = e^{-bx}$ and Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

(or)

Evaluate Parseval's Identity $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

proof: Consider $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

$$F_c[s] = F_c[f(x)] = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$$

$$F_c[g(x)] = F_c[s] = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+s^2}$$



we know that

$$\int_0^{\infty} F_c[f(x)] F_c[g(x)] ds = \int_0^{\infty} f(x) \cdot g(x) dx$$

$$\int_0^{\infty} \sqrt{\frac{a}{\pi}} \frac{a}{a^2+s^2} \sqrt{\frac{b}{\pi}} \frac{b}{b^2+s^2} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$$

$$\frac{2ab}{\pi} \int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= \frac{-1}{a+b} [0 - 1]$$

$$= \frac{1}{a+b}$$

$$\int_0^{\infty} \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{\pi}{2ab(a+b)}$$

2]. Evaluate $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx$ using Transform.

Soln.: consider $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$

$$F_s[f(x)] = F_s[e^{-ax}] = \sqrt{\frac{a}{\pi}} \frac{a}{s^2+a^2}$$

$$F_s[g(x)] = F_s[e^{-bx}] = \sqrt{\frac{b}{\pi}} \frac{b}{s^2+b^2}$$

Now,

$$\int_0^{\infty} F_s[f(x)] F_s[g(x)] ds = \int_0^{\infty} f(x) \cdot g(x) dx$$



$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+a^2} \sqrt{\frac{2}{\pi}} \frac{s}{s^2+b^2} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds = \int_0^{\infty} e^{-(a+b)x} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds = \left[\frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= -\frac{1}{(a+b)} [0 - 1]$$

$$\int_0^{\infty} \frac{s^2}{(s^2+a^2)(s^2+b^2)} ds = \frac{\pi}{2(a+b)}$$

$$\Rightarrow \int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{2(a+b)}$$

Parseval's Identity :

1]. Using transform methods, evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

Soln. :

consider $f(x) = e^{-ax}$

$$F_c [f(x)] = F_c [e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

$$\text{Now, } \int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_c(s)|^2 ds$$

$$\int_0^{\infty} e^{-2ax} dx = \int_0^{\infty} \frac{2}{\pi} \frac{a^2}{(s^2+a^2)^2} ds$$

$$\left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty} = \frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2+a^2)^2}$$



$$\frac{-1}{2a} [0 - 1] = \frac{2a^2}{\pi} \int_0^{\infty} \frac{ds}{(s^2 + a^2)^2}$$

$$\frac{\pi}{2a(2a^2)} = \int_0^{\infty} \frac{ds}{(s^2 + a^2)^2}$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$$

2]. Using transform methods, evaluate

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)^2} \text{ where } a > 0.$$

Soln.:

consider $f(x) = e^{-ax}$

$$F_s[f(x)] = F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2}$$

Now,

$$\int_0^{\infty} |f(x)|^2 dx = \int_0^{\infty} |F_s(s)|^2 ds$$

$$\int_0^{\infty} e^{-2ax} dx = \int_0^{\infty} \frac{2}{\pi} \frac{s^2}{(s^2 + a^2)^2} ds$$

$$\left[\frac{e^{-2ax}}{-2a} \right]_0^{\infty} = \frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(s^2 + a^2)^2} ds$$

$$\frac{-1}{2a} [0 - 1] = \frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(s^2 + a^2)^2} ds$$

$$\Rightarrow \int_0^{\infty} \frac{s^2}{(s^2 + a^2)^2} ds = \frac{\pi}{4a} \Rightarrow \int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx = \frac{\pi}{4a}$$