Problems on $(-\pi, \pi)$ and $(-l, l)$

1. Odd function:

If $f(x)$ is said to be odd, then

$f(-x) = -f(x)$.

Eg: $f(x) = x$ in $(-\pi, \pi)$

Now $f(-x) = -x = -f(x) \Rightarrow f(x)$ is odd.

Egs: x^3, $\sin x$, $\tan^3 x$

2. Even function:

If $f(x)$ is said to be even, then

$f(-x) = f(x)$.

Eg: $f(x) = x^2$, $\cos x$, $\sin^2 x$, $|x|$, $|\sin x|$, $x \sin x$.

Formula.

$f(x)$	$(-\pi, \pi)$	$(-l, l)$
odd	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$. Here $a_0 = 0$ and $a_n = 0$	$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$, $b_n = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l} dx$. Here $a_0 = 0$ and $a_n = 0$
even	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x) dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$. Here $b_n = 0$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$, $a_0 = \frac{2}{l}\int_0^{l} f(x) dx$, $a_n = \frac{2}{l}\int_0^{l} f(x) \cos \frac{n\pi x}{l} dx$. Here $b_n = 0$
Neither odd nor even	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}]$, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x) dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$, $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx$

Note:

Odd function × odd function = even function

even × even = even

even × odd = odd

odd × even = odd

① I. Find the fourier series for $f(x) = |x|$, $(-\pi, \pi)$

Soln:

$f(x) = |x|$ ② $f(x) = x, (-\pi, \pi)$

Now $f(-x) = |-x| = |x| = f(x)$

$\Rightarrow f(x)$ is an even function. Hence $b_n = 0$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx \quad \because f(x) = |x| = x, (0,\pi)$

$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$

$a_0 = \pi$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

$u = x$	$v = \cos nx$
$u' = 1$	$v_1 = \sin nx/n$
$u'' = 0$	$v_2 = -\cos nx/n^2$

$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2} + 0\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$

$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right] = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$

$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right) + 0\right]_0^{\pi}$

$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n} - 0\right]$

$b_n = -\frac{2(-1)^n}{n}$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \cos nx$$

$$= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^2} \cos nx$$

2]. Find the fourier series for $f(x) = |\cos x|, (-\pi,\pi)$

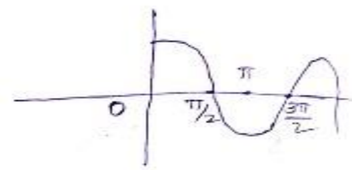
Soln. :

$$f(x) = |\cos x| = \begin{cases} \cos x, & 0 < x < \pi/2 \\ -\cos x, & \pi/2 < x < \pi \end{cases}$$

Now $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\Rightarrow f(x)$ is even. $\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$



$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx\right]$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[(1-0) - (0-1)\right]$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\, dx - \int_{\pi/2}^{\pi} \cos x \cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos nx \cos x\, dx - \int_{\pi/2}^{\pi} \cos nx \cos x\, dx\right]$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi/2} \left(\cos(nx+x) + \cos(nx-x)\right) dx - \int_{\pi/2}^{\pi} \left(\cos(nx+x) + \cos(nx-x)\right) dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}[\cos(n+1)x + \cos(n-1)x]\,dx - \int_{\pi/2}^{\pi}[\cos(n+1)x + \cos(n-1)x]\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$\sin(n-1)\pi/2 = \sin(n\pi/2 - \pi/2) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2} = -\cos\frac{n\pi}{2}$$

$$\sin(n+1)\pi/2 = \sin(n\pi/2 + \pi/2) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2} + \cos\frac{n\pi}{2}\sin\frac{\pi}{2} = \cos\frac{n\pi}{2}$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right) + \left(\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right)\right]$$

$$= \frac{\cos n\pi/2}{\pi}\left[\frac{2}{n+1} - \frac{2}{n-1}\right] = \frac{2\cos\frac{n\pi}{2}}{\pi}\left(\frac{n-1-n-1}{n^2-1}\right)$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \dfrac{-4}{\pi(n^2-1)}\cos\dfrac{n\pi}{2} & \text{if } n \text{ is even}\end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{2\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi/2 - (\pi - \pi/2)\right]$$

$$a_1 = 0$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=\text{even}}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}\cos nx$$

3. Find the F.S for $f(x) = x^2$, $(-\pi, \pi)$. Deduce that

i) $\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \cdots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$

Soln:

$f(x) = x^2$

Now, $f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx \cancel{+ b_n \sin nx}]$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$u = x^2$ | $v = \cos nx$

$u' = 2x$ | $v_1 = \frac{\sin nx}{n}$

$u'' = 2$ | $v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0$ | $v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) - 0\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx.$$

i) Put $x = \pi$ is a point of discontinuity.

$$\therefore f(x) = \frac{f(-\pi) + f(\pi)}{2} = \frac{(-\pi)^2 + \pi^2}{2} = \pi^2$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3(4)} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}$$

ii). Take $x=0$ is a point of continuity.

$$\therefore f(0) = 0$$

$$\therefore 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-\pi^2}{3(4)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots = -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

4. Find the fourier series $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$

Soln. :

$$f(x) = \begin{cases} \phi_1(x), & -\pi \le x \le 0 \\ \phi_2(x), & 0 \le x \le \pi \end{cases}$$

$$\phi_1(x) = 1 + \frac{2x}{\pi}; \qquad \phi_2(x) = 1 - \frac{2x}{\pi}$$

Now $\phi_1(-x) = 1 - \frac{2x}{\pi} = \phi_2(x)$

$\Rightarrow f(x)$ is an even. $\Rightarrow b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi} = \frac{2}{\pi}\left[(\pi - \pi) - 0\right]$$

$$a_0 = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx \, dx$$

$u = 1-\frac{2x}{\pi}$ | $v = \cos nx$

$u' = -2/\pi$ | $v_1 = \frac{\sin nx}{n}$

$u'' = 0$ | $v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2(-1)^n}{n^2\pi} + \frac{2}{n^2\pi}\right] = \frac{2}{\pi}\left(\frac{2}{n^2\pi}\right)\left[1-(-1)^n\right]$$

$$= \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \frac{8}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} \frac{8}{n^2\pi^2}\cos nx = \frac{8}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}\cos nx.$$

Take $x=0$ is a point of continuity.

$$\therefore f(0) = 1$$

$$1 = \frac{8}{\pi^2}\sum_{n=odd}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{8}$$

W. I. Find the Fourier Series for i) $f(x) = |\sin x|$, $(-\pi,\pi)$

ii). $f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$

iii) $f(x) = x+x^2$, $(-\pi,\pi)$