



UNIT-IV FOURIER SERIES AND FOURIER TRANSFORM FOURIER TRANSFORM-SELF RECIPROCAL FUNCTION

Problems on Self reciprocal function:

1. Show that the function $e^{-x^2/2}$ is self-reciprocal under Fourier transform.

Soln.:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2 - 2isx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[x^2 - 2isx + (is)^2 - (is)^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(x-is)^2 + s^2]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} e^{-\frac{s^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-is)^2}{2}} dx$$

$$\text{Put } t = \frac{x-is}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{2} dt$$

$$dt = \frac{dx}{\sqrt{2}}$$

$$dx = \sqrt{2} dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}} e^{-\frac{s^2}{2}} \sqrt{\pi}$$

$$\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$F(s) = e^{-s^2/2}$$



Hence

The Fourier transform of $e^{-x^2/2}$ is $e^{-s^2/2}$

Hence $e^{-x^2/2}$ is self-reciprocal under F.T.

27. How find the Fourier transform of $f(x) = e^{-x^2}$

28. Find the Fourier transform of $f(x) = e^{-a^2 x^2}$

Soln.:

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[a^2 x^2 - isx]} dx$$

$$2ax = b \\ b = \frac{is}{2a}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2 x^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2} dx$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$



$$\begin{aligned} \text{Put } t &= ax - \frac{is}{2a} & \left| \begin{array}{l} x = -\infty \Rightarrow t = -\infty \\ x = \infty \Rightarrow t = \infty \end{array} \right. \\ dt &= a dx \\ dx &= \frac{dt}{a} \\ &= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \\ &= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt \\ &= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \sqrt{\pi} \\ &= e^{-s^2/4a^2} \cdot \frac{1}{a\sqrt{2}} \end{aligned}$$