



UNIT 4

COMPLEX INTEGRATION

CAUCHY'S INTEGRAL THEOREM:

If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous with all points inside and on a simple closed curve C then $\int_C f(z) dz = 0$

CAUCHY'S INTEGRAL FORMULA:

If $f(z)$ is analytic inside and on a simple closed curve C and 'a' be any point inside C then $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, where the integration being taken in the anticlockwise direction around C .

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES:

If $f(z)$ is analytic inside and on a simple closed curve C and let 'a' be any point inside C then $\int_C \frac{f(z)}{(z-a)^2} dz = \frac{2\pi i f'(a)}{1!}$

$$\text{and } \int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i f''(a)}{2!}$$

In general,

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = 2\pi i f^{(n)}(a)$$

Solved Problems:

1. Evaluate $\int_C \frac{\cos \pi z}{z-1} dz$ where the circle C is $|z|=2$

Solution:



UNIT-4 **COMPLEX INTEGRATION**

Cauchy's integral formula

$$f(z) = \cos \pi z$$

$$z - 1 = 0$$

$$z = 1$$

$$\text{Given condition : } |z| = 2$$

i.e., $|1| < 2$ which lies inside C.

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\therefore \int_C \frac{\cos \pi z}{z-1} dz = 2\pi i f(1) = -2\pi i$$

2. Evaluate $\int_C \frac{dz}{z+2}$ where C is $|z|=1$

Solution:

$$f(z) = 1$$

$$z + 2 = 0$$

$$z = -2$$

$$\text{Given condition : } |z| = 1$$

i.e., $|-2| = 2 > 1$ which lies outside C.

Therefore by Cauchy's integral formula $\int_C \frac{dz}{z+2} = 0$

3. Evaluate $\int_C \frac{e^z}{z+1} dz$ where C is the circle $|z + \frac{1}{2}| = 1$

Solution:



$$f(z) = e^z$$

$$z + 1 = 0$$

$$z = -1$$

$$\text{Given condition : } \left| z + \frac{1}{2} \right| = 1$$

$$\text{i.e., } \left| -1 + \frac{1}{2} \right| = \frac{1}{2} < 1 \text{ which lies inside C.}$$

Therefore by Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\frac{e^z}{z+1} dz = 2\pi i f(-1)$$

$$= 2\pi i e^{-1}$$

$$= \frac{2\pi i}{e}$$

4. Using Cauchy's integral formula $\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is circle $|z|=1$ or $x^2+y^2=9$

Solution:

Consider

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \dots\dots\dots(1)$$

$$\frac{1}{(z-1)(z-2)} = \frac{A(z-2)}{z-1} + \frac{B(z-1)}{z-2}$$

$$1 = A(z-2) + B(z-1)$$

Substituting $z=2$,

$$1 = 0 + B \Rightarrow B = 1$$



Substituting $z=1$,

$$1=A(-1+0) \Rightarrow A=-1$$

Substituting $A=-1, B=1$ in (1)

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Multiply both sides by $\sin \pi z^2 + \cos \pi z^2$, we get

$$\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}$$

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = - \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$z-1=0 \quad \text{and} \quad z-2=0$$

$$z=1 \quad \quad \quad z=2$$

Given condition: $|z|=3$

i.e., $|1| < 3$ and $|2| < 3$ which lies inside C.

$$\therefore \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i (-1) + 2\pi i (2)$$

$$= 4\pi i \quad \quad \quad (\text{since } f(1)=-1 \text{ and } f(2)=1)$$

5. Evaluate $\int \frac{e^z}{(z-1)^3} dz$ where C is the circle $|z-1|=\frac{3}{2}$

Solution:



Substituting $z=1$,

$$1=A(-1+0) \Rightarrow A=-1$$

Substituting $A=-1, B=1$ in (1)

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

Multiply both sides by $\sin \pi z^2 + \cos \pi z^2$, we get

$$\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \frac{-(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}$$

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = - \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} dz + \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz$$

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

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$$z-1=0 \quad \text{and} \quad z-2=0$$

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Given condition: $|z|=3$

i.e., $|1| < 3$ and $|2| < 3$ which lies inside C.

$$\therefore \int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i (-1) + 2\pi i (2)$$

$$= 4\pi i \quad \quad \quad (\text{since } f(1)=-1 \text{ and } f(2)=1)$$

5. Evaluate $\int \frac{e^z}{(z-1)^3} dz$ where C is the circle $|z-1|=\frac{3}{2}$

Solution:



$$f(z) = e^z$$

$$z - 1 = 0$$

$$z = 1$$

$$\text{Given condition: } |z - 1| = \frac{3}{2}$$

$$\text{i.e., } |1 - 1| = 0 < \frac{3}{2} \text{ which lies inside } C.$$

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i f''(a)}{2!}$$

$$\text{Therefore } \int_C \frac{e^z}{(z-1)^3} dz = \frac{2\pi i f''(1)}{2!} = \frac{2\pi i e}{2} = \pi i e$$
