PROPERTIES:

Change of Scale property:

If $L\{f(t)\} = F(S)$, then

$$L[f(at)] = \frac{1}{a} F(S/a).$$

Proof:

We know that,

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$$

put $at = x \Rightarrow t = x/a$

$dt = dx/a$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(S/a)x} f(x)\, dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(S/a)t} f(t)\, dt$$

$$= \frac{1}{a} F(S/a)$$

First Shifting property:

If $L\{f(t)\} = F(s)$ then

(i) $L[e^{-at} f(t)] = \{L[f(t)]\}_{s \to s+a} = F(s+a)$

(ii) $L[e^{at} f(t)] = \{L[f(t)]\}_{s \to s-a} = F(s-a)$

Proof:

(i) We know that,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} [e^{-at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= F(s+a)$$

(ii) $$L[e^{at} f(t)] = \int_0^\infty e^{-st} [e^{at} f(t)]\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= F(s-a).$$

Second Shifting Property:

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L[g(t)] = e^{-as} F(s)$.

Proof:

$$L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$L[g(t)] = 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = u \Rightarrow dt = du$

When $t = a \Rightarrow u = 0$

$t \to \infty \Rightarrow u \to \infty$

$$L[g(t)] = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= \int_0^\infty e^{-us} e^{-as} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-us} f(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} f(t)\,dt \quad \text{Replace } u \to t$$

$$L[g(t)] = e^{-as} F(s).$$

Laplace transforms of derivatives:

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = s F(s) - f(0).$$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrating by parts we get,

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-s e^{-st})\,dt$$

$$= \left[e^{-\infty} f(\infty) - e^{0} f(0)\right] + s\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L\{f(t)\}$$

$$= s\,F(s) - f(0).$$

Corollary:

Let $f''(t) = s^2 F(s) - s f(0) - f'(0)$

Let $L[g'(t)] = s\,G(s) - g(0)$.

We know that,

$$L[f'(t)] = s\,L[f(t)] - f(0).$$

Replace $f(t) \to f'(t)$ & $f'(t) \to f''(t)$ & $f(0) \to f'(0)$

$$\Rightarrow L[f''(t)] = s\,L[f'(t)] - f'(0)$$

$$= s\left[s\left(L[f(t)] - f(0)\right)\right] - f'(0)$$

$$= s^2 L[f(t)] - s f(0) - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0).$$

Laplace Transform of integrals:

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \dfrac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)\,dt$ and $g(0) = 0$

then $g'(t) = f(t)$

WKT $L[g'(t)] = s\,L(g(t)) - g(0)$

$$= s\,L(g(t))$$

$$\Rightarrow L[g(t)] = \frac{1}{s} L[g'(t)]$$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s} L[f(t)]$ $\left\{ \because g(t) = \int_0^t f(t) \right.$
$g'(t) = f(t)$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $L[f(t)] = F(s)$ then

$L[t f(t)] = -\frac{d}{ds} F(s)$

Proof:

We know that,

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\,dt$$

$$= -L[t f(t)]$$

$$\Rightarrow L[t f(t)] = -\frac{d}{ds}[F(s)]$$

In general,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Problems :

Change of Scale property :

(1) Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

(2) Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}$$

Problems :

Change of Scale property :

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh t] = \frac{1}{s^2-1} = F(s)$$

$$L[\sinh 3t] = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 - 1}$$

$$= \frac{1}{3}\left(\frac{9}{s^2-9}\right)$$

$$= \frac{3}{s^2-9}$$

② Find $L(\cos 5t)$ using change of scale property ?

Soln:

$$L(\cos t) = \frac{s}{s^2+1} = F(s)$$

$$L(\cos 5t) = \frac{1}{5} F\left(\frac{s}{5}\right)$$

$$= \frac{1}{5}\left[\frac{s/5}{\left(\frac{s}{5}\right)^2+1}\right]$$

$$= \frac{1}{5}\left[\frac{5s}{s^2+25}\right]$$

$$= \frac{s}{s^2+25}$$

First Shifting theorem:

(1) Find $L[e^{-3t}\sin^2 t]$

Proof:

$$L[e^{-at}f(t)] = F(s+a)$$

$$L[e^{-3t}\sin^2 t] = L[\sin^2 t]_{s\to s+3}$$

$$= L\left[\frac{1-\cos 2t}{2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\{L(1) - L(\cos 2t)\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}_{s\to s+3}$$

$$= \frac{1}{2}\left\{\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right\}$$

$$= \frac{1}{2}\left\{\frac{4}{(s+3)[(s+3)^2+4]}\right\}$$

$$= \frac{2}{(s+3)[(s+3)^2+4]}$$

(2) Find $L(t^2 e^{-2t})$.

Soln:

$$L[e^{-at}f(t)] = F(s+a)$$

$$L[e^{-2t}t^2] = [L(t^2)]_{s\to s+2}$$

$$= \left[\frac{2}{s^3}\right]_{s\to s+2}$$

$$= \frac{2}{(s+2)^3}$$

(3) Find $L[e^{2t}\cos 5t]$.

Soln:

$$L[e^{2t}\cos 5t] = L[\cos 5t]_{s\to s-2}$$

$$= \left[\frac{s}{s^2+25}\right]_{s\to s-2}$$

$$= \frac{s-2}{(s-2)^2+25}$$

Second Shifting theorem:

(1) Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0<t<2 \\ 3, & t>2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_2^\infty e^{-st}\cdot 3\,dt$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{-3}{s}\left[e^{-\infty} - e^{-2s}\right]$$

$$= \frac{3e^{-2s}}{s}$$

(2) Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi \end{cases}$$

Soln:

$$\left[f(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & \pi<t<2\pi \end{cases} \right]$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi e^{-st} f(t)\,dt + \int_\pi^\infty e^{-st}\, 0\, dt$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi \qquad \left\{\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right\}$$

$$= \frac{e^{-s\pi}}{s^2+1}(-s\sin\pi - \cos\pi) + \frac{1}{s^2+1}$$

$$= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}$$

Laplace Transforms of Derivatives:

(1) Find $L[t\sin at]$

Soln:

$L[t\sin at] =$

$f(t) = t\sin at$

$f'(t) = at\cos at + \sin at$

$f''(t) = a[-at\sin at + \cos at] + a\cos at$

$= 2a\cos at - a^2 t\sin at$

$f(0) = 0, \quad f'(0) = 0$

$L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$

$L[2a\cos at - a^2 t\sin at] = s^2 L[t\sin at] - s(0) - 0$

$\Rightarrow 2a L(\cos at) - a^2 L(t\sin at) = s^2 L(t\sin at)$

$\Rightarrow (s^2+a^2) L(t\sin at) = 2a L(\cos at)$

$\Rightarrow (s^2+a^2) L(t\sin at) = 2a \cdot \frac{s}{a^2+s^2}$

$$L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$$

(4) Find $L\left[\frac{\sin 3t}{t}\right]$

Soln:

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty L[f(t)]\,ds$$

$$L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty L(\sin 3t)\,ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+9}\right) ds$$

$$= \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$= 3\cdot\frac{1}{3}\left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty \qquad \left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \pi/2 - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3).$$