



UNIT-V LAPLACE TRANSFORM

PROPERTIES:

Change of Scale property:

If $L\{f(t)\} = F(s)$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right). \quad (38 \text{ min.}) \text{ I bmt } (2)$$

Proof:

We know that, $e^{at} - e^{-at} = 2e^{at}$

$$L[f(at)] = \int_0^\infty e^{-st} f(at) dt$$

$$\text{Put } at = x \Rightarrow t = x/a$$

$$dt = dx/a \quad (38 \text{ min.}) \text{ I bmt } (2)$$

$$L[f(at)] = \int_0^\infty e^{-s(x/a)} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-s(x/a)} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x) dx \quad (38 \text{ min.}) \text{ I bmt } (3)$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t) dt$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right) \quad (38 \text{ min.}) \text{ I bmt } (3)$$



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First Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$(i) \mathcal{L}[e^{-at} f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s+a} = F(s+a)$$

$$(ii) \mathcal{L}[e^{at} f(t)] = \left\{ \mathcal{L}[f(t)] \right\}_{s \rightarrow s-a} = F(s-a)$$

Proof:

(i) We know that,

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$\mathcal{L}[e^{-at} f(t)] = \int_0^{\infty} e^{-st} [e^{-at} f(t)] dt$$

$$= \int_0^{\infty} e^{-(s+a)t} f(t) dt$$

$$= F(s+a)$$

$$(ii) \mathcal{L}[e^{at} f(t)] = \int_0^{\infty} e^{-st} [e^{at} f(t)] dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt$$

$$= F(s-a)$$

Second Shifting property:

If $\mathcal{L}\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t \geq a \\ 0, & t < a \end{cases}$
then $\mathcal{L}[g(t)] = e^{-as} F(s)$.

Proof:

$$\mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t) dt$$

$$= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt$$



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$$\begin{aligned} L[g(t)] &= 0 + \int_a^{\infty} e^{-st} f(t-a) dt \\ &= \int_a^{\infty} e^{-st} f(t-a) dt \\ \text{Put } t-a = u \Rightarrow dt = du \\ \text{When } t = a \Rightarrow u = 0 \\ t \rightarrow \infty \Rightarrow u \rightarrow \infty \\ L[g(t)] &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\ &= \int_0^{\infty} e^{-us} e^{-as} f(u) du \\ &= e^{-as} \int_0^{\infty} e^{-us} f(u) du \\ &= e^{-as} \int_0^{\infty} e^{-st} f(t) dt \quad \text{Replace } u \rightarrow t \\ L[g(t)] &= e^{-as} F(s) \end{aligned}$$

Laplace transforms of derivatives :

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = sF(s) - f(0).$$

Proof:

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

Integrating by parts we get,

$$= [e^{-st} f(t)]_0^{\infty} - \int_0^{\infty} f(t) (-se^{-st}) dt$$



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$$\begin{aligned} L[f(t)] &= \left[e^{-\infty} f(\infty) - e^0 f(0) \right] + s \int_0^\infty e^{-st} f(t) dt \\ (1) &= f(0) + s \int_0^\infty e^{-st} f(t) dt \\ &= s F(s) - f(0). \end{aligned}$$

Corollary:

Let $f''(t) = s^2 F(s) - s f(0) - f'(0)$

Let $L[g'(t)] = s G(s) - g(0)$

We know that,
 $L[f'(t)] = s L[f(t)] - f(0)$.

Replace $f(t) \rightarrow f'(t)$ & $f'(t) \rightarrow f''(t)$ &
 $f(0) \rightarrow f'(0)$

$$\begin{aligned} \Rightarrow L[f''(t)] &= s L[f'(t)] - f'(0) \\ &= s [s L[f(t)] - f(0)] - f'(0) \\ &= s^2 L[f(t)] - s f(0) - f'(0) \\ &= s^2 F(s) - s f(0) - f'(0). \end{aligned}$$

Laplace Transform of integrals:

If $L[f(t)] = F(s)$ then $L \left[\int_0^t f(t) dt \right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t) dt$ and $g(0) = 0$

then $g'(t) = f(t)$

WKT $L[g'(t)] = s L(g(t)) - g(0)$

$$\begin{aligned} &= s L(g(t)) \\ \Rightarrow L[g(t)] &= \frac{1}{s} L[g'(t)] \end{aligned}$$



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$$\Rightarrow L \left[\int_0^t f(t) dt \right] = \frac{1}{s} L [f(t)] \quad \left\{ \begin{array}{l} \because g(t) = \int_0^t f(t) dt \\ g'(t) = f(t) \end{array} \right.$$
$$\Rightarrow L \left[\int_0^t f(t) dt \right] = \frac{F(s)}{s}$$

Derivative of Laplace Transform (or) Laplace transform of $t f(t)$:

If $L[f(t)] = F(s)$ then

$$L[t f(t)] = - \frac{d}{ds} F(s)$$

Proof: We know that,

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\frac{d}{ds} F(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} (e^{-st}) f(t) dt$$

$$= \int_0^\infty -te^{-st} f(t) dt$$

~~Integrate by parts~~

$$= - \int_0^\infty e^{-st} t f(t) dt$$

~~Integrate by parts~~

$$= - L[t f(t)]$$

$$\Rightarrow L[t f(t)] = - \frac{d}{ds} [F(s)]$$

In general,

$$L[t^n f(t)] = (-i)^n \frac{d^n}{ds^n} [F(s)]$$



UNIT-V LAPLACE TRANSFORM

Problems : $\frac{1+2s^2}{(1-2s^2)(1+2s^2)} = [(\frac{1}{2}) + \frac{1}{2} \operatorname{coth}(\frac{s}{2})]$ (Ans)

Change of Scale property : $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$ (Ans) (applying scale property)

① Find $\mathcal{L}[\sinh 3t]$ by using change of scale property

Soln:

$$\mathcal{L}[\sinh ht] = \frac{1}{s^2 - h^2} F(s) \quad (1)$$
$$\mathcal{L}[\sinh 3t] = \frac{1}{3^2 - s^2} F\left(\frac{s}{3}\right) \quad (2)$$
$$= \frac{1}{9 - s^2} \frac{1}{s^2 - 1}$$
$$= \frac{1}{9 - s^2} \left[\frac{1}{s-1} - \frac{1}{s+1} \right]$$
$$= \frac{1}{9 - s^2} \left[\frac{1+2s^2}{1+2s^2} \right] \quad (Ans)$$

② Find $\mathcal{L}(\cos 5t)$ using change of scale property?

Soln:

$$\mathcal{L}(\cos t) = \frac{1}{s^2 + 1} F(s) \quad (1)$$

applying scale property $\mathcal{L}(\cos at) = \frac{1}{s^2 + a^2} F\left(\frac{s}{a}\right)$ (Ans)

$$\mathcal{L}(\cos 5t) = \frac{1}{5^2} F\left(\frac{s}{5}\right) \quad (2)$$
$$= \frac{1}{25} \left[\frac{1}{s^2 + 1} \right] \quad (Ans)$$
$$= \frac{1}{25} \left[\frac{1}{s^2 + 25} \right]$$
$$= \frac{s}{s^2 + 25} \cdot \frac{1}{25} = \frac{s}{25s^2 + 25}$$
$$= \frac{s}{25(s^2 + 1)} = \frac{1}{25} \cdot \frac{s}{s^2 + 1} \quad (Ans)$$



UNIT-V LAPLACE TRANSFORM

Problems : $\frac{1+2e^{-s}}{(1-2e^{-s})(1+2e^{-s})} = [1/(3+s)] + \text{invint}$ (Q)

Change of Scale property : $\text{Ans} \rightarrow \text{Ans} \times (\text{scale factor})^2$ (square)

① Find $L[\sinh 3t]$ by using change of scale property

Soln:

$$L[\sinh ht] = \frac{1}{(s-h)^2} = F(s) \quad \text{Ans.}$$
$$L[\sinh 3t] = \frac{1}{(s-3)^2} = \frac{1}{s^2-6s+9} = \frac{1}{[(s-3)^2-1]} = \frac{1}{(s-3)^2} \cdot \frac{1}{\frac{s^2-9}{(s-3)^2-1}} = \frac{1}{(s-3)^2} \cdot \frac{1}{\frac{1+2e^{-s}}{(1-2e^{-s})(1+2e^{-s})}} = \frac{1+2e^{-s}}{(1-2e^{-s})(1+2e^{-s})} \cdot \frac{1}{(s-3)^2} = \frac{1+2e^{-s}}{s^2-9} \cdot \frac{1}{(s-3)^2} = \frac{1+2e^{-s}}{(s-3)^2(s+3)^2}$$

② Find $L(\cos 5t)$ using change of scale property?

Soln:

$$L(\cos t) = \frac{1}{s^2+1} = F(s)$$

Integrating since $\int \cos t dt = \sin t + C$ Ans. + Invint (Q)

$$L(\cos 5t) = \frac{1}{s^2+25} = F\left(\frac{s}{5}\right) \quad \text{Ans.}$$
$$= \frac{1}{5} \left[\frac{s/5}{(s/5)^2+1} \right] = \frac{1}{5} \left[\frac{s/5}{s^2/25+1} \right] = \frac{1}{5} \left[\frac{s/5}{s^2+25} \right] = \frac{s/5}{s^2+25} \cdot \frac{1}{5} = \frac{s}{s^2+25} \cdot \frac{1}{25} = \frac{1}{25} \cdot \frac{s}{s^2+25}$$



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First Shifting theorem:

① Find $\mathcal{L}[e^{-3t} \sin^2 t] = []$

Proof:

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

$$\mathcal{L}[e^{-3t} \sin^2 t] = \frac{1}{2} \mathcal{L}[\sin^2 t]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left[\frac{1 - \cos 2t}{2} \right]_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ L(1) - L(\cos 2t) \right\}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s} - \left[\frac{s}{s^2 + 4} \right] \right\}_{s \rightarrow s+3}$$

$$= \frac{1}{2} \left\{ \frac{1}{s+3} - \frac{s+3}{(s+3)^2 + 4} \right\}$$

$$= \frac{1}{2} \left\{ \frac{4}{(s+3)[(s+3)^2 + 4]} \right\}$$

$$= \frac{2}{(s+3)[(s+3)^2 + 4]}$$

② Find $\mathcal{L}(t^2 e^{-2t})$.

Soln:

$$\mathcal{L}[e^{-at} f(t)] = F(s+a)$$

$$\mathcal{L}[e^{-2t} t^2] = [\mathcal{L}(t^2)]_{s \rightarrow s+2}$$

$$= \left[\frac{2}{s^3} \right]_{s \rightarrow s+2}$$

$$= \frac{2}{(s+2)^3}$$

$$= \frac{2}{(s+2)^3}$$



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Q) Find $L[e^{2t} \cos 5t]$

Soln:

$$L[e^{2t} \cos 5t] = L[\cos 5t]_{s \rightarrow s-2}$$

$$= \left[\frac{s}{s^2 + 25} \right]_{s \rightarrow s-2}$$

$$= \frac{s-2}{(s-2)^2 + 25}$$

Second Shifting theorem:

Q) Find $L[f(t)]$ where $f(t) = \begin{cases} 0, & 0 \leq t < 2 \\ 3, & t \geq 2 \end{cases}$

Soln:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^2 e^{-st} \cdot 0 dt + \int_2^\infty e^{-st} \cdot 3 dt$$

$$= 0 + \int_2^\infty e^{-st} \cdot 3 dt$$

$$= 3 \left[\frac{e^{-st}}{-s} \right]_2^\infty$$

$$= -3 \left[e^{-\infty} - e^{-2s} \right]$$

$$= \frac{3e^{-2s}}{s}$$

Q) Find the Laplace transform of $f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t > \pi \end{cases}$

Soln:

$$\{ f(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & \pi \leq t < 2\pi \end{cases} \}$$



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$$\begin{aligned}
 L[f(t)] &= \int_0^\infty e^{-st} f(t) dt \\
 &= \int_0^\pi e^{-st} f(t) dt + \int_\pi^\infty e^{-st} \cdot 0 dt \\
 &= \left[\frac{e^{-st}}{s^2+1} (-s \sin t - \cos t) \right]_0^\pi \quad \left\{ \begin{array}{l} \text{if } \int e^{ax} \sin bx dt \\ = -e^{ax} (a \sin bx - b \cos bx) \end{array} \right. \\
 &= \frac{e^{-\pi s}}{s^2+1} (-s \sin \pi - \cos \pi) + \frac{1}{s^2+1} \\
 &= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}
 \end{aligned}$$

Laplace Transforms of Derivatives:

① Find $L[t \sin at]$

Soln:

$$L[t \sin at] =$$

$$f(t) = t \sin at$$

$$f'(t) = at \cos at + \sin at$$

$$f''(t) = a[-at \sin at + \cos at] + a \cos at$$

$$(L \sin at) = a^2 a \cos at - [a^2 t \sin at]$$

$$f(0) = 0, f'(0) = 0$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[2a \cos at - a^2 t \sin at] = s^2 L[t \sin at] - s(0) - 0$$

$$\Rightarrow 2a L(\cos at) - a^2 L(t \sin at) = s^2 L(t \sin at)$$

$$\Rightarrow (s^2 + a^2) L(t \sin at) = 2a L(\cos at)$$

$$\Rightarrow (s^2 + a^2) L(t \sin at) = 2a \cdot \frac{s}{a^2 + s^2}$$

$$L(t \sin at) = \frac{2as}{(s^2 + a^2)^2}$$



UNIT-V LAPLACE TRANSFORM

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Find $\mathcal{L} \left[\frac{\sin 3t}{t} \right]$

Soln:

$$\mathcal{L} \left[\frac{f(t)}{t} \right] = \int_s^{\infty} F(s) ds = \int_s^{\infty} \frac{(s-2) s}{(s+2)(s-2)} \mathcal{L}[f(t)] ds$$

$$\mathcal{L} \left[\frac{\sin 3t}{t} \right] = \int_s^{\infty} \mathcal{L}[\sin 3t] ds, \quad \text{Ans} \quad (2)$$

$$= \int_s^{\infty} \left(\frac{3}{s^2 + 9} \right) ds$$

$$= \int_s^{\infty} \frac{3}{s^2 + 3^2} ds$$

$$= 3 \cdot \frac{1}{3} \left[\tan^{-1} \left(\frac{s}{3} \right) \right]_s^{\infty} \quad (\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right))$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/3)$$

$$= \pi/2 - \tan^{-1}(s/3)$$

$$= \cot^{-1}(s/3).$$