PERIODIC FUNCTIONS :

A funct. $f(t)$ is said to be periodic if $f(t+T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T) = f(t)$ for all t is called the period of the func.

Eg(1): The funct. $\sin t$ & $\cos t$ are periodic functions, both having period 2π.

$$\sin t = \sin(t+2\pi) = \sin(t+4\pi) = \ldots$$

Consider the func. $f(t) = \begin{cases} t & \text{if } 0<t<2 \\ 4-t & \text{if } 2<t<4 \end{cases}$ and $f(t+4) = f(t)$

$\therefore f(t)$ is a periodic func. with period 4.

LT of periodic functions:

Let $f(t)$ be a periodic function with period T. Then

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

① Find LT of $f(t) = \left(\frac{2t}{3}\right)$, $0 \le t \le 3$ & $f(t+3) = f(t)$

Soln: $f(t)$ is a periodic func. with period 3.

(ie) $T = 3$.

WKT $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-3s}} \int_0^3 e^{-st}\left(\frac{2t}{3}\right)dt = \frac{1}{1-e^{-3s}}\left(\frac{2}{3}\right)\int_0^3 e^{-st}\,t\,dt$$

$$= \frac{1}{1-e^{-3s}}\left(\frac{2}{3}\right)\left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{1}{1-e^{-3s}}\left(\frac{2}{3}\right)\left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-3s}}\left(\frac{2}{3}\right)\left[\frac{1-e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}\right]$$

② Find the LT of $f(t)$ if $f(t) = e^t$, $0 < t < 2\pi$ and $f(t) = f(t+2\pi)$

Soln: $f(t)$ is a periodic function with period 2π (ie) $T = 2\pi$

$$L[f(t)] = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} e^t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,dt = \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)t}}{(1-s)}\right]_0^{2\pi}$$

$$= \frac{e^{2\pi(1-s)} - 1}{(1-s)(1-e^{-2\pi s})}$$

3) Find LT of $f(t) = \begin{cases} t, & 0<t<1 \\ 2-t, & 1<t<2 \end{cases}$ such that $f(t+2) = f(t)$

Soln: $f(t)$ is a periodic func. with period 2 (i.e.) $T=2$.

$$L[f(t)] = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left\{ \int_0^1 e^{-st}\, t\, dt + \int_1^2 e^{-st}(2-t)\,dt \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{1-e^{-s}}{s^2} - \frac{e^{-s}}{s} + \frac{e^{-2s}-e^{-s}}{s^2} + \frac{e^{-s}}{s} \right\}$$

$$= \frac{1}{1-e^{-2s}} \left\{ \frac{1}{s^2} \left[e^{-2s} - 2e^{-s} + 1 \right] \right\}$$

$$= \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} \qquad \frac{(1-e^{-s})^2}{s\left[1-(e^{-s})^2\right]} = \frac{(1-e^{-s})^{\cancel{2}}}{s\cancel{(1-e^{-s})}(1+e^{-s})}$$

4) Find LT of the periodic func. $f(t) = \begin{cases} 1, & 0<t<a \\ -1, & a<t<2a \end{cases}$ & $f(t+2a) = f(t)$.

Soln: $$\frac{(1-e^{-sa})^2}{s(1-e^{-2as})}$$